

TWISTED COXETER ELEMENTS AND FOLDED AR-QUIVERS VIA DYNKIN DIAGRAM AUTOMORPHISMS: II

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ABSTRACT. As a continuation of the paper [24], we find a combinatorial interpretation of Dorey's rule for type C_n via twisted Auslander-Reiten quivers (AR-quivers) of type D_{n+1} , which are combinatorial AR-quivers related to certain Dynkin diagram automorphisms. Combinatorial properties of twisted AR-quivers are useful to understand not only Dorey's rule but also other notions in the representation theory of the quantum affine algebra $U'_q(C_n^{(1)})$ such as denominator formulas. In addition, unlike twisted adapted classes of type A_{2n-1} in [24], we show twisted AR-quivers of type D_{n+1} consist of the cluster point called twisted adapted cluster point. Hence, by introducing new combinatorial objects called twisted Dynkin quivers of type D_{n+1} , we give one to one correspondences between twisted Coxeter elements, twisted adapted classes and twisted AR-quivers.

1. INTRODUCTION

In [8], Dorey described relations between three-point couplings in the simply-laced affine Toda field theories (ATFTs) and Lie theories. More precisely, simply-laced ATFTs on untwisted affine Lie algebras are related to the same type of Lie algebras and simply-laced ATFTs on twisted affine Lie algebras are related to non-simply laced Lie algebras obtained by corresponding Dynkin diagram automorphisms.

Generally, in ATFTs, quantum affine algebras appear as quantum symmetry groups and the fundamental representations of quantum affine algebras correspond to the quantum particles in the theories [5]. In particular, the Dorey's rule was interpreted by Chari-Pressley [7] in the language of representations of quantum affine algebras. For type A_n and D_n , they used representations associated to Coxeter elements and, for type B_n and C_n , they used representations associated to twisted Coxeter elements ([25]) of type A_{2n-1} and D_{n+1} . Recently, the first author [20, 21] found combinatorial interpretation of Dorey's rule for type A_n and D_n using Auslander-Reiten quivers ([2]) and, in [24], we generalized his work for type B_n using twisted Auslander-Reiten quivers (AR-quivers). In this paper, we introduce and investigate twisted AR-quivers of type D_{n+1} to find a combinatorial statement of Dorey's rule for type C_n .

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A twisted AR-quiver is in the center of this paper and it is a generalization of an AR-quiver which gives an answer to Question 1.1. For any given Dynkin quiver Q of finite type $AD E_n$, the AR-quiver Γ_Q can be understood as a realization of the convex partial order \prec_Q on the set of positive roots Φ^+ ([4]): For $\alpha, \beta \in \Phi^+$,

$$\alpha \prec_Q \beta \Leftrightarrow \text{there is a path from } \beta \text{ to } \alpha \text{ in } \Gamma_Q.$$

The set of all AR-quivers Γ_Q has one to one correspondences with the set of convex partial orders \prec_Q , Dynkin quivers Q , Coxeter elements ϕ_Q and classes $[Q]$ of reduced expressions of the longest element $w_0 \in W$ adapted to Q whose cardinalities are 2^{n-1} . Moreover, the set of $[Q]$'s can be grouped into the *adapted cluster point* $\llbracket Q \rrbracket$; that is

$$(1.1) \quad \begin{array}{ccccc} & & \{\prec_Q\} & & \\ & \swarrow^{1-1} & \updownarrow^{1-1} & \searrow^{1-1} & \\ \{[Q]\} & \xleftarrow{1-1} & \{\phi_Q\} & \xrightarrow{1-1} & \{Q\} \\ & \swarrow^{1-1} & \updownarrow^{1-1} & \searrow^{1-1} & \\ & & \{\Gamma_Q\} & & \end{array} \quad \text{for } 2^{n-1}\text{-many Dynkin quivers } Q.$$

For a Weyl group W of finite type, a combinatorial AR-quiver $\Upsilon_{[\tilde{w}]}$ associated to a class of reduced expressions $[\tilde{w}]$ of $w \in W$ was introduced in [23] as a generalization of AR-quiver in the sense that $\Upsilon_{[\tilde{w}]}$ is a realization of the partial order $\prec_{[\tilde{w}]}$ on $\Phi(w) = \{\alpha \in \Phi^+ | w^{-1}\alpha \in -\Phi^+\}$ (see Remark 2.17 for the definition of $\prec_{[\tilde{w}]}$).

Since Dorey's rules for type C_n are described via *twisted Coxeter elements* of D_{n+1} , we reach to the following natural question.

Question 1.1. Is there any non-adapted cluster point which is associated to twisted Coxeter elements? For the cluster point, can we generalize the correspondences in (1.1)?

Twisted Coxeter elements of type D_{n+1} depend on the automorphisms (1.2) and (1.3) of Dynkin diagrams. In Section 7 and Appendix A, we give an affirmative answer to Question 1.1 for the both types of twisted Coxeter elements associated to (1.2) and (1.3).

$$(1.2) \quad C_n \ (n \geq 3) \longleftrightarrow \left(D_{n+1} : \begin{array}{c} \text{Diagram: } \textcircled{1} \text{---} \textcircled{2} \cdots \textcircled{n-1} \begin{array}{l} \nearrow \textcircled{n} \\ \searrow \textcircled{n+1} \end{array} \\ \text{Labels: } 1 \quad 2 \quad \dots \quad n-1 \quad n \quad n+1 \end{array}, \ i^{\vee(n+1,2)} = \begin{cases} i & \text{if } i \leq n-1, \\ n+1 & \text{if } i = n, \\ n & \text{if } i = n+1. \end{cases} \right)$$

$$(1.3) \quad G_2 \longleftrightarrow \left(D_4 : \begin{array}{c} \text{Diagram: } \textcircled{1} \text{---} \textcircled{4} \begin{array}{l} \nearrow \textcircled{2} \\ \searrow \textcircled{3} \end{array} \\ \text{Labels: } 1 \quad 4 \quad 2 \quad 3 \end{array}, \ \begin{cases} 1^{\vee(4,3)} = 2, \ 2^{\vee(4,3)} = 3, \ 3^{\vee(4,3)} = 1, \\ 4^{\vee(4,3)} = 4. \end{cases} \right)$$

In Section 3, we construct a commutation class of reduced expressions $[\tilde{w}_0]$ of w_0 for each twisted Coxeter element and show such classes form a cluster point $\llbracket Q^\leftarrow \rrbracket$, called *the twisted cluster point*. Moreover, in Section 7, we introduce new combinatorial model Q^\leftarrow , called a *twisted Dynkin quiver of type D_{n+1}* and prove that

$$\text{a reduced expression } \tilde{w}_0 \text{ is in } [Q^\leftarrow] \in \llbracket Q^\leftarrow \rrbracket \text{ if and only if } \tilde{w}_0 \text{ is adapted to } Q^\leftarrow$$

(see Definition 7.7). Note that twisted Dynkin quivers of type D_{n+1} can be understood as *oriented Dynkin diagrams of type C_n* (see Remark 7.2). As a consequence, the twisted

analogue of (1.1) holds:

$$(1.4) \quad \begin{array}{ccccc} & & \{ \prec_{[Q^{\leftarrow}]} \} & & \\ & \swarrow 1-1 & \updownarrow 1-1 & \nwarrow 1-1 & \\ \{ [Q^{\leftarrow}] \} & \xleftarrow{1-1} & \{ \phi_{Q^{\leftarrow}} \} & \xrightarrow{1-1} & \{ Q^{\leftarrow} \} \\ & \searrow 1-1 & \downarrow 1-1 & \swarrow 1-1 & \\ & & \{ \Upsilon_{[Q^{\leftarrow}]} \} & & \end{array} \quad \text{for } 2^n\text{-many twisted Dynkin quivers } Q^{\leftarrow}\text{'s.}$$

Here, $\Upsilon_{[Q^{\leftarrow}]}$ is a *twisted AR-quiver* equipped with a coordinate system. Remark that for the twisted adapted classes of A_{2n-1} type, (1.4) is not true.

Another goal of this paper is answering to the following question.

Question 1.2. Is there a combinatorial way to find labels of twisted AR-quiver $\Upsilon_{[Q^{\leftarrow}]}$?

In [20, 21], the first named author showed a purely combinatorial way to find labels of Γ_Q in Φ^+ . Analogous to the adapted cases, in Section 4 and 5, we describe purely combinatorial ways to find labels of $\Upsilon_{[Q^{\leftarrow}]}$. In order to do this, we introduce new quivers with coordinate systems, called *folded AR-quivers* $\hat{\Upsilon}_{[Q^{\leftarrow}]}$.

An interesting result in Section 4 is that each twisted adapted class $[Q^{\leftarrow}]$ of D_{n+1} is closely related to the adapted class $[Q]$ where Q is the associated Dynkin quiver of type A_n . Moreover, a folded AR-quiver $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ of D_{n+1} can be obtained by *gluing* two copies of AR quiver $\Gamma_{[Q]}$ of A_n .

On the other hand, in Section 5, we introduce another method to find labels of twisted AR-quiver $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ using *swings* (see Definition 5.7 for swings.) This result implies a folded AR-quiver $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ of type D_{n+1} has similar properties with a full subquiver of $\Gamma_{[Q]}$ of *type* D_{n+2} consisting of vertices with residues 1 to n (see [21, Theorem 1.20] and (9.4) also).

Finally, using the combinatorial properties of folded AR-quivers, we give answers to the following question.

Question 1.3. Are there applications of twisted and folded AR-quivers to the representation theory of $U'_q(C_n^{(1)})$? Can we derive Dorey's rule for type C_n from $\hat{\Upsilon}_{[Q^{\leftarrow}]}$?

In [22], using the properties of Γ_Q discovered in [20, 21], the first named author introduced *generalized distances* $\text{gdist}_{[\tilde{w}_0]}$ of a sequence of positive root lattice elements, *radiuses* $\text{rds}_{[\tilde{w}_0]}$ of roots and *socles* $\text{soc}_{[\tilde{w}_0]}$ of pairs in Φ^+ associated to any $[\tilde{w}_0]$. He proved that (i) the Dorey's rule for $U'_q(A_n^{(1)})$ (resp. $U'_q(D_n^{(1)})$) can be interpreted as the coordinates of $(\alpha, \beta, \gamma) \in (\Phi^+)^3$ in Γ_Q where $\alpha + \beta = \gamma \in \Phi^+$, (ii) denominator formulas $d_{k,l}(z)$ for $U'_q(A_n^{(1)})$ (resp. $U'_q(D_n^{(1)})$) can be read from *any* Γ_Q , (iii) $\text{rds}_{[Q]}(\gamma) = \mathbf{m}(\gamma)$, $\text{gdist}_{[Q]}(\alpha, \beta) \leq \max\{\mathbf{m}(\alpha), \mathbf{m}(\beta)\}$ and the notion $\text{soc}_{[Q]}$ is well-defined. Here, $\mathbf{m}(\alpha)$ denotes the multiplicity of α (see Definition 2.16 for the multiplicity).

As an answer to Question 1.3, in Section 8 and Section 9, we generalize (i)~(iii) by introducing *folded multiplicities* and applying the combinatorial properties of $\hat{\Upsilon}_{[Q^{\leftarrow}]}$: (i') the Dorey's rule for $U'_q(C_n^{(1)})$ can be interpreted as the coordinates of $(\alpha, \beta, \gamma) \in (\Phi_{D_{n+1}}^+)^3$ in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$

where (α, β) is a $[Q^\leftarrow]$ -minimal pair for γ , (ii') denominator formulas $d_{k,l}(z)$ for $U'_q(C_n^{(1)})$ can be read from any $\hat{\Upsilon}_{[Q^\leftarrow]}$, (iii') $\text{rds}_{[Q^\leftarrow]}(\gamma) = \overline{\mathbf{m}}(\gamma)$, $\text{gdist}_{[Q^\leftarrow]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\alpha), \overline{\mathbf{m}}(\beta)\}$ and the notion $\text{soc}_{[Q^\leftarrow]}$ is well-defined. Here $\overline{\mathbf{m}}(\alpha)$ denotes the *folded multiplicity* of α (see Definition 4.28).

Note that generalized Cartan matrices of $D_{n+1}^{(2)}$ and $C_n^{(1)}$ are transpose to each other. As we mentioned in [24], such relation was investigated in the theory of representations of quantum groups via q -characters in [9, 11]. Moreover, in the forthcoming paper by Kashiwara-Oh [16], the categorical relations between representations of $U_q(D_{n+1}^{(2)})$ and $U_q(C_n^{(1)})$ will be described using the results in this paper.

2. BACKGROUNDS

2.1. Foldable r -cluster point and related reduced expressions. In this subsection, we recall the definition of foldable r -cluster points of type ADE and related notions and properties. For more detail, see [24].

Let $I_n = \{1, 2, \dots, n\}$ be the index set of Dynkin diagram Δ_n^X of finite type X_n . Denote by W_n^X and ${}_n w_0^X$ the Weyl group and the longest element. The Weyl group W_n^X is generated by simple reflections $\{s_i \mid i \in I_n\}$. We sometimes omit the type and the rank if there is no danger of confusion.

For $w \in W$, let i_1, i_2, \dots, i_l be indices satisfying $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. If there is no sequence $j_1, j_1, \dots, j_{l'}$ of indices such that $w = s_{j_1} s_{j_2} \cdots s_{j_{l'}}$ and $l' < l$ then we say l is the *length* of w and denote by $\ell(w)$. The sequences $\mathbf{i} := i_1 i_2 \cdots i_l$ (resp. $s_{i_1} s_{i_2} \cdots s_{i_l}$) is a *reduced word* (resp. *reduced expression*) of w . We often abuse the notation \mathbf{i} , instead of the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_l}$. Also, note that we denote by \mathbf{i}_0 a reduced word of the longest element $w_0 \in W$ and by ℓ the length of w_0 .

Definition 2.1. Let \mathbf{i} and \mathbf{j} be reduced expressions of $w \in W$. If \mathbf{j} can be obtained by applying commutation relations, $s_i s_j = s_j s_i$, to \mathbf{i} then we say \mathbf{i} and \mathbf{j} are *commutation equivalent* and denote $\mathbf{i} \sim \mathbf{j}$. The *commutation class* of \mathbf{i} is denoted by $[\mathbf{i}]$.

It can be easily shown that if $\mathbf{j} \sim \mathbf{i}$ and \mathbf{i} is a reduced expression of w then \mathbf{j} is also a reduced word of $w \in W$. Now, if $w = w_0$, there is another way to find reduced expressions from a given reduced expression \mathbf{i}_0 by Proposition 2.2. (See [23], for example.)

Proposition 2.2. Let $* : I \rightarrow I, i \mapsto i^*$, be the involution induced by w_0 on I (See [6]). For a reduced word $\mathbf{i}_0 = i_1 i_2 \cdots i_\ell$ of w_0 , two other words $\mathbf{i}'_0 = i_2 i_3 \cdots i_\ell i_1^*$ and $\mathbf{i}''_0 = i_\ell^* i_1 i_2 \cdots i_{\ell-1}$ are also reduced words of w_0 . Moreover, we have $[\mathbf{i}_0] \neq [\mathbf{i}'_0], [\mathbf{i}''_0]$.

Definition 2.3. [23] Let \mathbf{i}_0 be a reduced expression of w_0 .

- (1) If there is a reduced expression $\mathbf{i}'_0 \in [\mathbf{i}_0]$ (resp. $\mathbf{i}''_0 \in [\mathbf{i}_0]$) such that $\mathbf{i}'_0 = i_1 i_2 i_3 \cdots i_\ell$ (resp. $\mathbf{i}''_0 = i_1 i_2 i_3 \cdots i_{\ell-1} i$) then the index i is called a *sink* (resp. *source*) of $[\mathbf{i}_0]$.
- (2) The right and left *reflection functors* r_i for $i \in I$ are defined as follows:

$$[\mathbf{i}_0] r_i = \begin{cases} [i_2 i_3 \cdots i_\ell i_1^*] & \text{if } i = i_1 \text{ is a sink of } [\mathbf{i}_0] \text{ and } i_1 i_2 \cdots i_\ell \in [\mathbf{i}_0], \\ [\mathbf{i}_0] & \text{otherwise;} \end{cases}$$

$$r_i[\mathbf{i}_0] = \begin{cases} [\mathbf{i}_\ell^* i_1 i_2 i_3 \cdots i_{\ell-1}] & \text{if } i = i_\ell \text{ is a source of } [\mathbf{i}_0] \text{ and } i_1 i_2 \cdots i_\ell \in [\mathbf{i}_0], \\ [\mathbf{i}_0] & \text{otherwise.} \end{cases}$$

For a sequence of indices $\mathbf{i} := j_1 j_2 \cdots j_t$, we denote by $[\mathbf{i}_0] r_{\mathbf{i}} := [\mathbf{i}_0] r_{j_1} r_{j_2} \cdots r_{j_t}$ and $r_{\mathbf{i}}[\mathbf{i}_0] := r_{j_t} r_{j_{t-1}} \cdots r_{j_1}[\mathbf{i}_0]$.

- (3) If $[\mathbf{i}'_0]$ can be obtained by applying reflection functors to $[\mathbf{i}_0]$ then we say $[\mathbf{i}_0]$ and $[\mathbf{i}'_0]$ are *reflection equivalent* and denote $[\mathbf{i}_0] \stackrel{r}{\sim} [\mathbf{i}'_0]$. The family of commutation classes

$$[[\mathbf{i}_0]] := \{ [\mathbf{i}'_0] \mid [\mathbf{i}_0] \stackrel{r}{\sim} [\mathbf{i}'_0] \}$$

is called an *r-cluster point*.

By the definition of reflection functors, we can see that if $[\mathbf{i}_0] \stackrel{r}{\sim} [\mathbf{i}'_0]$ for $\mathbf{i}_0 = i_1 i_2 \cdots i_\ell$ and $\mathbf{i}'_0 = i'_1 i'_2 \cdots i'_\ell$ then

$$\#\{i_s \mid s = 1, \dots, \ell \text{ such that } i_s = i \text{ or } i^*\} = \#\{i'_s \mid s = 1, \dots, \ell \text{ such that } i'_s = i \text{ or } i^*\}$$

for any $i \in I$. Hence we consider the following notion introduced in [24].

Definition 2.4. [24]

- (1) Let $\vee : I \rightarrow I$ be an automorphism such that $i \mapsto i^\vee$. We say \vee is *compatible* with the involution $*$ if i and i^* are in the same orbit class \bar{i} determined by \vee .
- (2) Let \vee be an automorphism on I compatible with $*$ and $\mathbf{i}_0 = i_1 i_2 \cdots i_\ell$ be a reduced expression of w_0 . Denote by $\bar{I} = \{\bar{i} \mid i \in I\}$ the set of orbits determined by \vee . Then the \vee -Coxeter composition is

$$\mathbf{C}_{[\mathbf{i}_0]} := (\mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{j}_1), \mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{j}_2), \dots, \mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{j}_p)), \quad p = |\bar{I}|,$$

where the smallest representative of \bar{j}_r is less than that of \bar{j}_s if and only if $r < s$ and

$$\mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{k}) := |\{i_s \mid i_s \in \bar{k}, 1 \leq s \leq \ell\}|.$$

Due to the compatibility of \vee in Definition 2.4, if $[\mathbf{i}_0]$ and $[\mathbf{i}'_0]$ are in the same r-cluster point then they have the same \vee -Coxeter composition. Hence we can denote the \vee -Coxeter composition associated to $[\mathbf{i}_0]$ by $\mathbf{C}_{[\mathbf{i}_0]}^\vee$.

Definition 2.5. [24, Definition 1.11] For a given automorphism \vee , the r-cluster point $[[\mathbf{i}_0]]$ is said to be \vee -foldable if

$$\mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{j}_r) = \mathbf{C}_{[\mathbf{i}_0]}^\vee(\bar{j}_s)$$

for any $\bar{j}_r, \bar{j}_s \in \bar{I}$.

Convention 2.6. If there is no danger of confusion, we identify the index \bar{j}_s in \bar{I} with $s \in \mathbb{Z}$. Hence the sum (resp. subtraction) $\bar{j}_s + \bar{j}_r$ (resp. $\bar{j}_s - \bar{j}_r$) is considered as that in \mathbb{Z} so that it is $s + r$ (resp. $s - r$) and, similarly, $\bar{j}_s \pm t := s \pm t$ for any $t \in \mathbb{Z}$.

In this paper, we deal with \vee -foldable cluster points for \vee in (1.2). The following remark shows the \vee -Coxeter composition of such cluster points.

Remark 2.7. Let \vee be defined in (1.2). If $[\![i_0]\!]$ is \vee -foldable, then

$$(2.1) \quad C_{[\![i_0]\!]}^\vee = \underbrace{(n+1, \dots, n+1)}_{n\text{-times}}.$$

Let \vee be defined in (1.3). If $[\![i_0]\!]$ is \vee -foldable or \vee^2 -foldable, then

$$(2.2) \quad C_{[\![i_0]\!]}^\vee = (6, 6).$$

Now let us recall another important notion called a *twisted Coxeter element* (resp. a *triply twisted Coxeter element*). Let $\sigma \in GL(\mathbb{C}\Phi)$ be a linear transformation of finite order which preserves the set of simple roots $\Pi \subset \Phi$ and consider $W\sigma \subset GL(\mathbb{C}\Phi)$. Note that the Weyl group W acts on $W\sigma$ by conjugations.

Definition 2.8.

- (1) Let $\Pi := \bigsqcup_{t=1}^k \Pi_{i_t}$, where Π_{i_t} are orbits by the automorphism σ . Take $s_{i_t} \in \Pi_{i_t}$ for $t = 1, \dots, k$ and consider the element $w = s_{i_{\tau(1)}} s_{i_{\tau(2)}} \cdots s_{i_{\tau(k)}} \in W$ for a permutation $\tau \in \mathfrak{S}_k$. Then the element $w\sigma \in W\sigma$ is called a σ -Coxeter element.
- (2) If σ in (1) is \vee in (1.2) then σ -Coxeter element is also called a *twisted Coxeter element* of type D_{n+1} .
- (3) If σ in (1) is \vee in (1.3) then σ -Coxeter element is also called a *triply twisted Coxeter element* of type D_4 .
- (4) If σ is the identity map, a σ -Coxeter element is called a *Coxeter element*.

We remark some properties which are useful in the following sections.

Remark 2.9. Note that w in (1) of Definition 2.8 is *fully commutative*, i.e. reduced expressions of w consist of the unique commutation class.

Proposition 2.10. [24, Proposition 2.6] *The number of twisted Coxeter elements associated to (1.2) is 2^n .*

2.2. (Combinatorial) Auslander-Reiten quivers. For type ADE , there are one-to-one correspondences between the set of commutation classes of adapted reduced expressions of w_0 , Dynkin quivers, Coxeter elements and Auslander-Reiten quivers. In this section, we recall the correspondences and introduce the notion of combinatorial Auslander-Reiten quivers, which is a generalization of Auslander-Reiten quivers in a combinatorial aspect. Details can be found in [2, 3, 10, 20, 22, 23].

Let Δ be a Dynkin diagram of type ADE_n and Q be a Dynkin quiver, which is obtained by assigning direction to every edge in Δ . The set of positive roots is denoted by Φ^+ and $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots. Gabriel theorem (Theorem 2.11) and the construction of Auslander-Reiten quiver (Definition 2.14) shows the relation between quiver representations and positive roots.

Theorem 2.11 (Gabriel's theorem). [10] *There is the bijection between the set $\text{Ind}Q$ of isomorphism classes of finite dimensional indecomposable modules over the path algebra $\mathbb{C}Q$ and Φ^+ which takes a class to the corresponding dimension vector.*

- Definition 2.12.** (1) An index $i \in I$ is called a *sink* (resp. *source*) of Q if there is no arrow exiting from (resp. entering to) the vertex i .
- (2) Denote by $s_i Q$ or $i Q$ the quiver obtained by reversing every arrow connecting i and another vertex if i is a sink or a source. Otherwise, we let $s_i Q = i Q = Q$.
- (3) Let $\mathbf{i} = i_1 i_2 \cdots i_{\ell(w)}$ be a reduced expression of w . We say \mathbf{i} is *adapted to the Dynkin quiver* Q if i_k is a sink of $i_{k-1} i_{k-2} \cdots i_2 i_1 Q$ for $k = 1, \dots, \ell(w)$.

For adapted reduced expressions of w_0 , there are well-known facts which is stated below.

Theorem 2.13.

- (1) *There is the natural one-to-one correspondence between the set of commutation classes of adapted reduced expressions of w_0 and the set of Dynkin quivers. Hence we denote by $[Q]$ the class of reduced expressions of w_0 which are adapted to the quiver Q .*
- (2) *There exists a unique Coxeter element ϕ_Q which is adapted to Q . Conversely, any Coxeter element is adapted to some Dynkin quiver Q .*

Hence there are canonical one-to-one correspondences between rank 2^{n-1} sets:

$$(2.3) \quad \{[Q]\} \xleftrightarrow{1-1} \{\phi_Q\} \xleftrightarrow{1-1} \{Q\}$$

Note that there exists a unique cluster point consisting of all adapted classes $[Q]$. we call it the *adapted cluster point* and denote by $\llbracket Q \rrbracket$.

Definition 2.14. For a Dynkin quiver Q of finite type ADE , let us take a reduced word \mathbf{i}_0 in $[Q]$. The quiver $\Gamma_Q = (\Gamma_Q^0, \Gamma_Q^1)$ is called the *Auslander-Reiten quiver* (AR-quiver) if

- (1) each vertex in Γ_Q^0 corresponds to an isomorphism class $[M]$ in $\text{Ind} Q$,
- (2) an arrow $[M] \rightarrow [M']$ in Γ_Q^1 represents an irreducible morphism $M \rightarrow M'$.

By Gabriel's theorem, every vertex of an AR-quiver can be labeled by a positive root and there is a well-known combinatorial construction of an AR-quiver. In order to introduce its simpler construction and properties, let us consider the subset

$$\Phi(w) := \{ \beta \in \Phi^+ \mid w^{-1}(\beta) \in -\Phi^+ \} = \{ \beta_k^{\mathbf{i}} \mid \beta_k^{\mathbf{i}} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}), k = 1, \dots, \ell(w) \}.$$

Note that $|\Phi(w)| = \ell(w)$ and $\Phi(w_0) = \Phi^+$.

Definition 2.15. [13, §5.3] For a positive root $\alpha = \sum_{i \in I} m_i \alpha_i \in \Phi^+$, the *support of α* is denoted by $\text{supp}(\alpha)$ and is defined by

$$\text{supp}(\alpha) := \{ \alpha_k \mid m_k \neq 0, k \in I \}.$$

Also, if $\alpha_k \in \text{supp}(\alpha)$ then we say α_k is a support of α .

Definition 2.16. For any $\gamma \in \Phi^+$, the *multiplicity of γ* , denoted by $\mathbf{m}(\gamma)$ is the positive integer defined as follows:

$$\mathbf{m}(\gamma) = \max \{ m_i \mid \sum_{i \in I} m_i \alpha_i = \gamma \}.$$

If $\mathbf{m}(\gamma) = 1$, we say that γ is *multiplicity free*.

Remark 2.17. Let \mathbf{i} be a reduced word of w . The *total order* $<_{\mathbf{i}}$ on $\Phi(w)$ is defined by

$$\beta_k^{\mathbf{i}} <_{\mathbf{i}} \beta_l^{\mathbf{i}} \text{ if and only if } k < l.$$

Then $<_{\mathbf{i}}$ is *convex*, i.e., if $\alpha, \beta, \alpha + \beta \in \Phi^+$ and $\alpha <_{\mathbf{i}} \beta$ then $\alpha <_{\mathbf{i}} \alpha + \beta <_{\mathbf{i}} \beta$. Using the convex total orders $<_{\mathbf{i}}$, the *convex partial order* $\prec_{[\mathbf{i}]}$ can be defined as follows:

$$\alpha \prec_{[\mathbf{i}]} \beta \iff \alpha <_{\mathbf{i}'} \beta \text{ for any } \mathbf{i}' \in [\mathbf{i}].$$

If \mathbf{i}_0 is adapted to Q then we denote $\prec_{[\mathbf{i}_0]}$ by \prec_Q and \prec_Q is closely related to the AR-quiver.

Definition 2.18. A combinatorial way to construct the AR-quiver Γ_Q with a Coxeter element ϕ_Q is given as follows ([12, §2.2]):

- (1) Let us denote $\phi_Q = s_{i_1} s_{i_2} \cdots s_{i_n}$ and consider a *height function* $\xi : I \rightarrow \mathbb{Z}$ satisfying $\xi(i) = \xi(j) + 1$ if there is an arrow from j to i in Q . Take the injection $\tilde{\phi}_Q : \Phi(\phi_Q) \rightarrow I \times \mathbb{Z}$ such that $\beta_k^{\phi_Q} := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \mapsto (i_k, \xi(i_k))$.
- (2) Inductively, we extend $\tilde{\phi}_Q$ to the map on Φ^+ satisfying that if $\beta, \phi_Q(\beta) \in \Phi^+$ and $\tilde{\phi}_Q(\beta) = (i, p)$ then $\tilde{\phi}_Q(\phi_Q(\beta)) = (i, p - 2)$.
- (3) If (i, p) and (j, q) are in $\text{Im}(\tilde{\phi}_Q)$, two indices i and j are connected in Δ and $q = p + 1$ then there is an arrow $(i, p) \rightarrow (j, q)$.

Note that if $\tilde{\phi}_Q(\alpha) = (i, p)$ for $\alpha \in \Phi^+$, then we say that i is the *residue of α with respect to Q* or α is in the i -th residue in Γ_Q . Also, Γ_Q has the *additive property*, i.e. if $\tilde{\phi}_Q(\beta) = (i, p)$ then

$$(2.4) \quad \beta + \phi_Q(\beta) = \sum_{\tilde{\phi}_Q(\gamma) = (j, p-1)} \gamma,$$

where i and j are adjacent in Q . Also, the followings are useful properties of Γ_Q .

Proposition 2.19. [4, 10, 27] *Let us denote by h^\vee the dual Coxeter number associated to Q . In Γ_Q , there are $r_i^Q = (h^\vee + a_i^Q - b_i^Q)/2$ vertices in the i -th residue, where a_i^Q (resp. b_i^Q) is the number of arrows in Q between i and i^* directed toward i (resp. i^*).*

Thus, for all $i \in I$, we have

$$(2.5) \quad r_i^Q + r_{i^*}^Q = n + 1 \quad \text{if } Q \text{ is of type } A_n.$$

Proposition 2.20. [4, 23, 26]

- (1) For distinct Dynkin quivers Q and Q' , we have $\Gamma_Q \not\cong \Gamma_{Q'}$ as quivers.
- (2) For $\alpha, \beta \in \Phi^+$, we have $\alpha \prec_Q \beta$ if and only if there is a path from β to α in Γ_Q .

Hence we get the following diagram [24]:

$$(2.6) \quad \begin{array}{ccccc} & & \{\prec_Q\} & & \\ & \swarrow^{1-1} & \updownarrow^{1-1} & \nwarrow^{1-1} & \\ \{[Q]\} & \xleftarrow{1-1} & \{\phi_Q\} & \xrightarrow{1-1} & \{Q\} \text{ for } [Q] \in \llbracket Q \rrbracket. \\ & \searrow_{1-1} & \updownarrow_{1-1} & \swarrow_{1-1} & \\ & & \{\Gamma_Q\} & & \end{array}$$

We close this subsection introducing the algorithm to get Γ_{iQ} from Γ_Q for a sink i of Q .

Algorithm 2.21. Let h^\vee be the dual Coxeter number associated to Q .

- (A1) Remove the vertex (i, p) such that $\tilde{\phi}_Q(\alpha_i) = (i, p)$ and the arrows entering into (i, p) .
- (A2) Add a vertex $(i^*, p - h^\vee)$ and arrows to $(j, p - h^\vee + 1)$ for all j adjacent to i^* in Δ .
- (A3) Label the vertex $(i^*, p - h^\vee)$ with α_i and substitute the labels β with $s_i(\beta)$ for all $\beta \in \Phi^+ \setminus \{\alpha_i\}$.

In [23], the authors constructed the combinatorial AR-quivers $\Upsilon_{[i_0]}$ for any commutation class $[i_0]$ of any finite type, which can be understood as a generalization of AR-quivers.

Algorithm 2.22. Let $\mathbf{i}_0 = (i_1 i_2 i_3 \cdots i_N)$ be a reduced expression of an element $w_0 \in W$. The quiver $\Upsilon_{\mathbf{i}_0} = (\Upsilon_{\mathbf{i}_0}^0, \Upsilon_{\mathbf{i}_0}^1)$ associated to \mathbf{i}_0 is constructed in the following algorithm:

- (Q1) $\Upsilon_{\mathbf{i}_0}^0$ consists of N vertices labeled by $\beta_1^{\mathbf{i}_0}, \dots, \beta_N^{\mathbf{i}_0}$.
- (Q2) The quiver $\Upsilon_{\mathbf{i}_0}$ consists of $|I|$ residues and each vertex $\beta_k^{\mathbf{i}_0} \in \Upsilon_{\mathbf{i}_0}^0$ lies in the i_k -th residue.
- (Q3) There is an arrow from $\beta_k^{\mathbf{i}_0}$ to $\beta_j^{\mathbf{i}_0}$ if the followings hold:
 - (Ar1) two vertices i_k and i_j are connected in the Dynkin diagram,
 - (Ar2) $j = \max\{j' \mid j' < k, i_{j'} = i_j\}$,
 - (Ar3) $k = \min\{k' \mid k' > j, i_{k'} = i_k\}$.
- (Q4) Assign the color $m_{jk} = -(\alpha_{i_j}, \alpha_{i_k})$ to each arrow $\beta_k^{\mathbf{i}_0} \rightarrow \beta_j^{\mathbf{i}_0}$ in (Q3); that is, $\beta_k^{\tilde{w}} \xrightarrow{m_{jk}} \beta_j^{\tilde{w}}$. Replace $\xrightarrow{1}$ by \rightarrow , $\xrightarrow{2}$ by \Rightarrow and $\xrightarrow{3}$ by \Rightarrow .

By the following theorem, a combinatorial AR-quiver can be understood as a generalization of an AR-quiver which proposes the generalization of Proposition 2.20. (See [23].) Also, we have a simple algorithm to get $\Upsilon_{[i_0]r_i}$ from $\Upsilon_{[i_0]}$ for a sink i of $[i_0]$.

- Algorithm 2.23.**
- (A'1) Remove the vertex v_0 of $\Upsilon_{[i_0]}$ in the i -th residue which does not have arrows exiting from v_0 and remove every arrow entering into v_0 .
 - (A'2) Add a vertex v_1 in the i^* -th residue and add arrows from v_1 to a vertex v such that
 - (i) v is in the j -th residue for an adjacent vertex j to i^* in Δ ,
 - (ii) $v' \prec_{[i_0]} v$ for any other vertex v' in the i^* -th or j -th residue.
 - (A'3) The label of v_1 is α_i . For the other vertices, substitute the label α by $s_i(\alpha)$.

Theorem 2.24. [23] Let us choose any commutation class $[i_0]$ of w_0 and a reduced word \mathbf{i}_0 in $[i_0]$.

- (1) The construction of $\Upsilon_{\mathbf{i}_0}$ does depend only on its commutation class $[\mathbf{i}_0]$ and hence $\Upsilon_{[\mathbf{i}_0]}$ is well-defined.
- (2) $\alpha \prec_{[\mathbf{i}_0]} \beta$ if and only if there exists a path from β to α in $\Upsilon_{[\mathbf{i}_0]}$.
- (3) By defining the notion, standard tableaux of shape $\Upsilon_{[\mathbf{i}_0]}$, every reduced word $\mathbf{i}'_0 \in [\mathbf{i}_0]$ corresponds to a standard tableau of shape $\Upsilon_{[\mathbf{i}_0]}$ and can be obtained by reading residues in a way compatible with the tableaux.
- (4) When $[\mathbf{i}_0] = [Q]$, $\Upsilon_{[Q]}$ is isomorphic to Γ_Q as quivers.

3. TWISTED ADAPTED CLUSTER POINT OF TYPE D_{n+1}

3.1. Twisted adapted cluster point and twisted Coxeter element. In this section, we denote by Δ , W and Φ^+ the Dynkin diagram, the Weyl group and the set of positive roots of type D_{n+1} . Also, we let \vee be the automorphism in (1.2). By identifying $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($1 \leq i \leq n$) and $\alpha_{n+1} = \varepsilon_n + \varepsilon_{n+1}$, positive roots in Φ^+ can be denoted as follows:

$$\Phi_{D_{n+1}}^+ = \{ \langle a_1, -a_2 \rangle, \langle b_1, b_2 \rangle, \langle c, n+1 \rangle \mid 1 \leq a_1 < a_2 \leq n+1, 1 \leq b_1 < b_2 \leq n, 1 \leq c \leq n \}$$

where

$$(3.1) \quad \begin{cases} \langle a_1, -a_2 \rangle = \sum_{i=a_1}^{a_2-1} \alpha_i = \varepsilon_{a_1} - \varepsilon_{a_2}, \\ \langle b_1, b_2 \rangle = \sum_{i=b_1}^{b_2-1} \alpha_i + 2 \sum_{j=b_2}^{n-1} \alpha_j + \alpha_n + \alpha_{n+1} = \varepsilon_{b_1} + \varepsilon_{b_2}, \\ \langle c, n+1 \rangle = \sum_{i=c}^{n+1} \alpha_i - \alpha_n = \varepsilon_c + \varepsilon_{n+1}. \end{cases}$$

Recall that

$$\Phi_{A_n}^+ = \{ [a, b] := \sum_{k=a}^b \alpha_k = \varepsilon_a - \varepsilon_{b+1} \mid 1 \leq a \leq b \}.$$

In order to distinguish positive roots of type D_m and A_m , we use the following definition.

Definition 3.1.

- (1) For $\langle a, b \rangle \in \Phi_{D_m}^+$, we denote a and b by the *first and second summands* of $\alpha = \langle a, b \rangle$, respectively.
- (2) For $\beta = [a, b] \in \Phi_{A_m}^+$, we denote a and b by the *first and second components* of β , respectively. If $\beta = \alpha_a$ then we write β as $[a]$.

Consider the twisted Coxeter element $(s_1 s_2 \cdots s_n)^\vee$ of D_{n+1} . In this subsection, we denote by the reduced expression of w_0 (see Remark 3.2 below)

$$\mathbf{i}_0 = \prod_{k=0}^n (1 \ 2 \ \cdots \ n)^{k^\vee}$$

where

$$(3.2) \quad (j_1 \cdots j_n)^\vee := j_1^\vee \cdots j_n^\vee \text{ and } (j_1 \cdots j_n)^{k^\vee} := (\underbrace{\cdots ((j_1 \cdots j_n)^\vee)^\vee \cdots}_{k\text{-times}})^\vee.$$

Note that

$$(1) \ (1 \ 2 \ \cdots \ n)^{k^\vee} = \begin{cases} 1 \ 2 \ \cdots \ n-1 \ n & \text{if } k \text{ is even,} \\ 1 \ 2 \ \cdots \ n-1 \ n+1 & \text{if } k \text{ is odd,} \end{cases}$$

- (2) $\mathbf{N} := n(n+1) \in 2\mathbb{Z}_{\geq 1}$ is the cardinality of $|\Phi_{D_{n+1}}^+|$ and coincides with the 2 times of $\mathbf{n} := |\Phi_{A_n}^+| = n(n+1)/2$.
- (3) $[\mathbf{i}_0]$ is the r -cluster point of \mathbf{i}_0 .

We can check that \mathbf{i}_0 is a reduced expression of w_0 by direct computations.

Remark 3.2. By the definition of \vee , if we denote

$$\beta_{p,q}^{\mathbf{i}_0} = \prod_{k=0}^{p-2} (s_1 s_2 \cdots s_n)^{k\vee} (s_1 s_2 \cdots s_{q-1})^{(p-1)\vee} (\alpha_{q(p-1)\vee}) \text{ for } p \in \{1, \dots, n+1\}, q = \{1, \dots, n\}$$

then $\beta_{1,q}^{\mathbf{i}_0} = \langle 1, -q-1 \rangle$, $\beta_{n+1,q}^{\mathbf{i}_0} = \langle q, n+1 \rangle$ and for $2 \leq p \leq n$

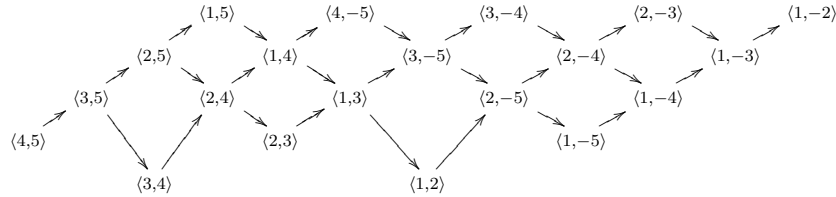
$$\beta_{p,q}^{\mathbf{i}_0} = \begin{cases} \langle p, -q-p \rangle & \text{if } p+q \leq n+1, \\ \langle p+q-n-1, p \rangle & \text{if } p+q > n+1. \end{cases}$$

Since $\{\beta_{p,q}^{\mathbf{i}_0}\} = \Phi^+$, the word \mathbf{i}_0 is a reduced expression of w_0 . Note that $[\mathbf{i}_0]$ is \vee -foldable and is not adapted to any Dynkin quiver Q of type D_{n+1} . We also note that

$$(3.3) \quad \prod_{k=0}^n (1 \ 2 \ \cdots \ n-1 \ n+1)^{k\vee}$$

is a reduced expression of w_0 .

Example 3.3. The following is the combinatorial AR-quiver of $[\mathbf{i}_0]$ of type D_5 .



Definition 3.4.

- (1) The cluster point $[\mathbf{i}_0]$ is called *the twisted adapted cluster point* of type D_{n+1} .
- (2) A class $[\mathbf{i}'_0] \in [\mathbf{i}_0]$ is called a *twisted adapted class* of type D_{n+1} .

Consider the map

$$\mathfrak{p}_{A_n}^{D_{n+1}} : \{ \text{twisted Coxeter elements of } D_{n+1} \} \rightarrow \{ \text{Coxeter elements of } A_n \}$$

such that $[i_1 \ i_2 \ \cdots \ i_n] \vee \mapsto \begin{cases} i_1 \ i_2 \ \cdots \ i_n & \text{if } i_t = n, \\ (i_1 \ i_2 \ \cdots \ i_n)^\vee & \text{if } i_t = n+1, \end{cases} \text{ for } t \text{ such that } i_t \in \{n, n+1\}.$

Proposition 3.5. *The map $\mathfrak{p}_{A_n}^{D_{n+1}}$ is a two-to-one and onto map.*

Proof. Suppose $i_1 \ i_2 \ \cdots \ i_n$ is a Coxeter element of A_n . Then both $[i_1 \ i_2 \ \cdots \ i_n] \vee$ and $[(i_1 \ i_2 \ \cdots \ i_n)^\vee] \vee$ are twisted Coxeter elements of D_{n+1} . Hence we proved the proposition. \square

Proposition 3.6. *For a twisted Coxeter element $[s_{i_1}s_{i_2}\cdots s_{i_n}]\vee$ of D_{n+1} , the word*

$$\prod_{i=0}^n (i_1 \ i_2 \ \cdots \ i_n)^{k\vee}$$

is a reduced expression of w_0 .

Proof. Assume that $\mathbf{i}'_0 = \prod_{i=0}^n (i_1 \ i_2 \ \cdots \ i_n)^{k\vee}$ is reduced. Then it is easy to check that

$$(3.4) \quad \left[\prod_{i=0}^n (i_2 \ \cdots \ i_n \ i_1^\vee)^{k\vee} \right] \text{ is also reduced.}$$

By Proposition 3.5, our assertion follows from the fact that any Dynkin quiver Q of type A_n can be written as follows:

- Let us denote by $\overleftarrow{Q} : \circ_1 \xleftarrow{\quad} \circ_2 \xleftarrow{\quad} \cdots \xleftarrow{\quad} \circ_{n-1} \xleftarrow{\quad} \circ_n$.
- $Q = i_k \cdots i_1 \overleftarrow{Q}$ for some $k \in \mathbb{Z}_{\geq 0}$ such that i_k is a sink of the quiver $i_{s-1} \cdots i_1 \overleftarrow{Q}$ ($1 \leq s \leq k$).

□

Lemma 3.7. *Let $\mathbf{i}'_0 = \prod_{i=0}^n (i_1 \ i_2 \ \cdots \ i_n)^{k\vee}$ where $[s_{i_1}s_{i_2}\cdots s_{i_n}]\vee$ is a twisted Coxeter element of D_{n+1} . If i is a sink of $[\mathbf{i}'_0]$ then there is a reduced expression $\mathbf{j} = j_1 j_2 \cdots j_n$ such that*

- $[j_1 \ j_2 \ \cdots \ j_n] = [i_1 \ i_2 \ \cdots \ i_n]$,
- $j_1 = i$,
- $[\mathbf{i}'_0] = [\prod_{i=0}^n (j_1 \ j_2 \ \cdots \ j_n)^{k\vee}]$

Proof. The assertion follows from the fact that $s_{i_1}s_{i_2}\cdots s_{i_n}$ is fully commutative. □

Proposition 3.8. *Let $[\mathbf{i}'_0]$ be a twisted adapted class of type D_{n+1} . Then there is a twisted Coxeter element $[i_1 \ i_2 \ \cdots \ i_n]\vee$ such that*

$$[\mathbf{i}'_0] = \left[\prod_{i=0}^n (i_1 \ i_2 \ \cdots \ i_n)^{k\vee} \right].$$

Proof. The proof is an immediate consequence of previous lemma. □

Remark 3.9. By Proposition 3.6 and Proposition 3.8, we can consider $\mathbf{p}_{A_n}^{D_{n+1}}$ as a two-to-one map between twisted adapted classes of type D_{n+1} and adapted classes of type A_n , i.e., $[[\mathbf{i}_0]] \twoheadrightarrow [[Q]]$. Thus, from now on, we abuse the notation $\mathbf{p}_{A_n}^{D_{n+1}}$ for the twisted adapted classes of D_{n+1} .

Theorem 3.10.

- (1) *There is the natural one-to-one correspondence between twisted Coxeter elements and twisted adapted classes of type D_{n+1} .*
- (2) *Since the number of adapted classes of type A_n is 2^{n-1} , the number of classes in the twisted adapted cluster point of type D_{n+1} is 2^n .*

Remark 3.11. In the twisted adapted cluster point of type A_{2n+1} [24], there are some classes which are not associated to twisted Coxeter elements. The number of twisted adapted classes of type A_{2n+1} is 2^{2n} and the number of twisted Coxeter element of type A_{2n+1} is $3^{n-1} \cdot 4$. Hence Theorem 3.10 is a special property for the type D_{n+1} case.

4. TWISTED AND FOLDED AR-QUIVERS AND THEIR LABELING VIA AR-QUIVERS OF TYPE A_n

4.1. AR-quivvers of type A_n . In this subsection, we briefly review the combinatorial properties of AR-quiver for a Dynkin quiver Q of type A_n , which were studied in [20].

Definition 4.1. [20, Definition 1.6] Fix any class $[j_0]$ of w_0 of any finite type.

- (a) A path in $\Upsilon_{[j_0]}$ is *N-sectional* (resp. *S-sectional*) if it is a concatenation of upward arrows (resp. downward arrows).
- (b) An *N-sectional* (resp. *S-sectional*) path ρ is *maximal* if there is no longer *N-sectional* (resp. *S-sectional*) path containing ρ .
- (c) For a sectional path ρ , the *length* of ρ is the number of all arrows in ρ .

Proposition 4.2. [23, Proposition 4.5] Fix any class $[j_0]$ of w_0 of type A_n . Let ρ be an *N-sectional* (resp. *S-sectional*) path in $\Upsilon_{[j_0]}$. Then every positive roots contained in ρ has the same first (resp. second) component.

Theorem 4.3. [20, Corollary 1.12] Fix any Dynkin quiver Q of type A_n . For $1 \leq i \leq n$, the AR quiver Γ_Q contains a maximal *N-sectional* path of length $n - i$ once and exactly once whose vertices share i as the first component. At the same time, Γ_Q contains a maximal *S-sectional* path of length $i - 1$ once and exactly once whose vertices share i as the second component.

Thus, for any AR-quiver Γ_Q of type A_n , we say that an *N-sectional* path ρ is the *maximal N-sectional path of length k* ($0 \leq k \leq n - 1$) if all positive roots contained in ρ have $n - k$ as first components. Similarly, we can define a notion of the *maximal S-sectional path of length k* ($0 \leq k \leq n - 1$) whose positive roots contain $k + 1$ as second components. Note that, in this paper, we only need maximal sectional paths (see Section 5).

Lemma 4.4. [14, Lemma 3.2.3] For any Dynkin quiver Q of type AD_m , the positions of simple roots inside of Γ_Q are on the boundary of Γ_Q ; that is, either

- (i) α_i is a sink or a source of Γ_Q , or
- (ii) residue of α_i is $\begin{cases} 1 \text{ or } n & \text{if } Q \text{ is of type } A_n, \\ 1, n \text{ or } n + 1 & \text{if } Q \text{ is of type } D_{n+1}. \end{cases}$

Furthermore,

- (a) all sinks and sources of Γ_Q have their labels as simple roots,
- (b) i is a sink (resp. source) of Q if and only if α_i is a sink (resp. source) of Γ_Q

Definition 4.5. If $[i_0]$ is a twisted adapted class then the combinatorial AR-quiver $\Upsilon_{[i_0]}$ is called a *twisted AR-quiver*.

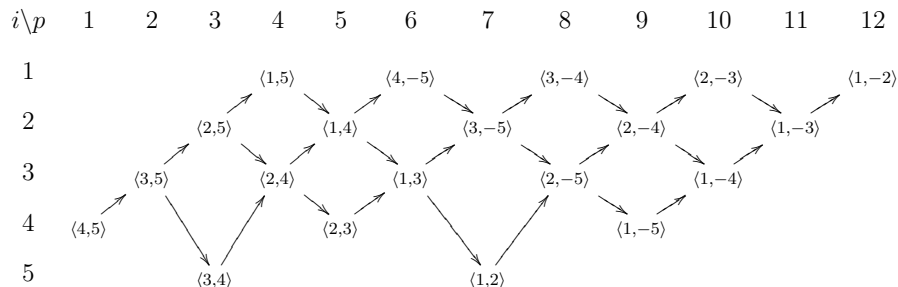
$$(4.1) \quad \mathfrak{B}_{[i_0]} := \Phi(\phi_{[i_0]}) = \{\beta_k^{\phi_{[i_0]}} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) \text{ for } k = 1, 2, \dots, n\}$$

Algorithm 4.6.

- (a) Let us define a height function $\xi : \bar{I}_{n+1}^D \rightarrow \mathbb{Z}$ where I_{n+1}^D is the index set of D_{n+1} and $\bar{I}_{n+1}^D = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ as follows : $\xi(\bar{i}) = \xi(\bar{j}) + 1$ if there is an arrow from j to i in Q . Note that ξ is unique up to constant.
- (b) Take the injection $\tilde{\phi}_{[\bar{i}_0]} : \mathfrak{B}_{[\bar{i}_0]} \rightarrow I \times \mathbb{Z}$ such that $\beta_k^{\phi_{[\bar{i}_0]}} \mapsto (i_k, \xi(\bar{i}_k))$.
- (c) Let us denote $\psi_{[\bar{i}_0]} = s_{i_1} s_{i_2} \cdots s_{i_n} s_{i_1^\vee} s_{i_2^\vee} \cdots s_{i_n^\vee}$ and $\beta_{n+k}^{\psi_{[\bar{i}_0]}} = \phi_{[\bar{i}_0]} s_{i_1^\vee} s_{i_2^\vee} \cdots s_{i_{k-1}^\vee}(\alpha_{i_k^\vee})$. We extend the map $\tilde{\phi}_{[\bar{i}_0]}$ to the map on $\Phi(\Psi_{[\bar{i}_0]})$ satisfying $\beta_{n+k}^{\psi_{[\bar{i}_0]}} \mapsto (i_k^\vee, \xi(\bar{i}_k) - 2)$.
- (d) Extend the map $\tilde{\phi}_{[\bar{i}_0]}$ to the map on Φ^+ satisfying if both β and $\psi_{[\bar{i}_0]}(\beta)$ are positive roots and $\tilde{\phi}_{[\bar{i}_0]}(\beta) = (i, p)$ then $\tilde{\phi}_{[\bar{i}_0]}(\psi_{[\bar{i}_0]}(\beta)) = (i, p - 4)$.
- (e) If (i) $(i, p), (j, q) \in \text{Im}(\tilde{\phi}_{[\bar{i}_0]})$, (ii) $|j - i| = 1$ or $(i, j) \in \{(n - 1, n + 1), (n + 1, n - 1)\}$, (iii) $q = p + 1$ then there is an arrow $(i, p) \rightarrow (j, q)$.

for each $\beta \in \Phi^+$ and $[\mathbf{i}_0]$ of w_0 , the residue i of β with respect to $[\mathbf{i}_0]$ is well-assigned.

Example 4.8. The twisted AR-quiver $\Upsilon_{[i_0]}$ for $i_0 = \prod_{k=0}^4 (4\,3\,2\,1)^{k^\vee}$ with coordinates can be depicted as follows:



Here ξ is defined by $\xi(\bar{1}) = 12$, $\xi(\bar{2}) = 11$, $\xi(\bar{3}) = 10$ and $\xi(\bar{4}) = 9$, and $\mathfrak{p}_{A_n}^{D_{n+1}}([\mathbf{i}_0]) = [Q]$ where

$$Q = \begin{array}{c} \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \\ 1 \qquad 2 \qquad 3 \qquad 4 \end{array}.$$

The labels on $(k, \xi(\bar{k}))$ and $(k^\vee, \xi(\bar{k}) - 2)$ for $k = 1, 2, 3, 4$ are determined by (b) and (c), respectively, in Algorithm 4.6 and the rest of labels are determined by (d).

In the sense of [24, Section 7], we introduce *the folded Auslander-Reiten quiver* $\hat{\Upsilon}_{[\mathbf{i}_0]}$ of a twisted adapted class $[\mathbf{i}_0]$:

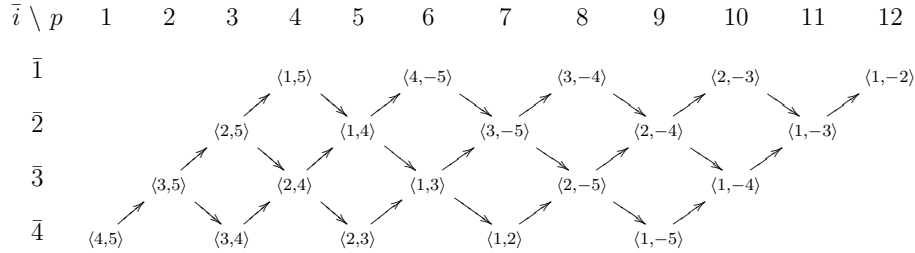
- (1) In the folded AR-quiver, we assign *folded coordinate* $(\bar{i}, p, (-1)^{\delta_{i,n+1}}) \in \bar{I} \times \mathbb{Z} \times \{\pm 1\}$ for the positive root β such that $\tilde{\phi}_{[\mathbf{i}_0]}(\beta) = (i, p)$.
- (2) There is an arrow from $(\bar{i}, p, (-1)^{\delta_{i,n+1}}) \rightarrow (\bar{j}, q, (-1)^{\delta_{j,n+1}})$ if and only if there is an arrow $(i, p) \rightarrow (j, q)$ in $\Upsilon_{[\mathbf{i}_0]}$.

In $\hat{\Upsilon}_{[\mathbf{i}_0]}$, we often omit the third coordinate and use only the first and second coordinates, (\bar{i}, p) . It is not hard to see that the set of coordinates which are assigned to positive roots is

$$(4.2) \quad \bar{\mathcal{I}} := \{(\bar{i}, \xi(\bar{i}) - 2t) \mid t = 0, 1, \dots, n\}.$$

Also, there is an arrow $(\bar{i}, p) \rightarrow (\bar{j}, q)$ for $(\bar{i}, p), (\bar{j}, q) \in \bar{\mathcal{I}}$ if and only if there are $i \in \bar{i}$ and $j \in \bar{j}$ such that $|\bar{i} - \bar{j}| = 1$ and $q = p + 1$. We call \bar{i} *the folded residue of β with respect to $[\mathbf{i}_0]$* when β is located at (\bar{i}, p) in $\hat{\Upsilon}_{[\mathbf{i}_0]}$.

Example 4.9. Let $[\mathbf{i}'_0]$ be the one in Example 4.8. Then the folded AR-quiver $\hat{\Upsilon}_{[\mathbf{i}'_0]}$ with coordinates is



Remark 4.10. Let $[\mathbf{i}_0]$ be a twisted adapted class with $\mathfrak{p}_{A_n}^{D_{n+1}}([\mathbf{i}_0]) = [Q]$. Then one can easily check that

- (i) the subquiver consisting of $\mathfrak{B}_{[\mathbf{i}_0]}$ inside $\hat{\Upsilon}_{[\mathbf{i}_0]}$ is isomorphic to Q as a Dynkin quiver of type A_n .

By (2.5) and (4.2), we can take

- (ii) the full subquiver ${}_1\Gamma_Q$ inside $\Upsilon_{[\mathbf{i}_0]}$ such that it contains $\mathfrak{B}_{[\mathbf{i}_0]}$ and is isomorphic to Γ_Q as quivers.

Since the subquiver consisting of leftmost vertices inside of Γ_Q is isomorphic to Q^* ,

- (iii) the full subquiver ${}_2\Gamma_{Q^*} := \Upsilon_{[\mathbf{i}_0]} \setminus {}_1\Gamma_Q$ is isomorphic to Γ_{Q^*} ,

where Q^* is a Dynkin quiver obtained by swapping vertices $i \leftrightarrow n+1-i$ and $\Upsilon_{[\mathbf{i}_0]} = {}_1\Gamma_Q \sqcup {}_2\Gamma_{Q^*}$ as sets of vertices.

4.3. Labeling of twisted and folded AR-quivers of D_{n+1} via AR-quivers of A_n .

Proposition 4.11. *Let $[\mathbf{i}_0]$ be a twisted adapted class. Suppose $\mathbf{j}_0 = j_1 j_2 \cdots j_N \in [\mathbf{i}_0]$ and let t_1 and t be the smallest and second smallest integers such that $j_{t_1}, j_t \in \bar{n}$. If $j_{t_1} = n$ (resp. $j_{t_1} = n+1$) and $\mathbf{p}_{A_n}^{D_{n+1}}([\mathbf{i}_0]) = [Q]$ then there is a reduced expression $\mathbf{k}_0 \in [Q]$ of A_n starting with $j_1 j_2 \cdots j_{t-1}$ (resp. $(j_1 j_2 \cdots j_{t-1})^\vee$)*

Proof. Note that if $j_{t_1} = n$ (resp. $j_{t_1} = n+1$), then $j_t = n+1$ (resp. $j_t = n$). By Theorem 2.24 and Remark 4.10, the assertion comes from the compatible readings of (i) \mathbf{j}_0 inside of ${}_1\Gamma_Q$ first, and then (ii) ${}_1\Gamma_Q \setminus \mathbf{j}_0$ inside of ${}_1\Gamma_Q$. Here, all vertices in the n -th and the $n+1$ -th-resides have to be read as n . \square

Remark 4.12. In the above proposition, we can assume that \mathbf{j}_0 is a reduced expression in $[\mathbf{i}_0]$ with the largest t without any loss of generality. By Theorem 2.24 and Theorem 4.3, such \mathbf{j}_0 can be obtained by the following way:

- (4.3) (i) Read the residues of all vertices in the right part of the length $n-1$ S -sectional path ρ inside ${}_1\Gamma_Q$. (ii) Read residues on the path ρ . (iii) Read the remained.

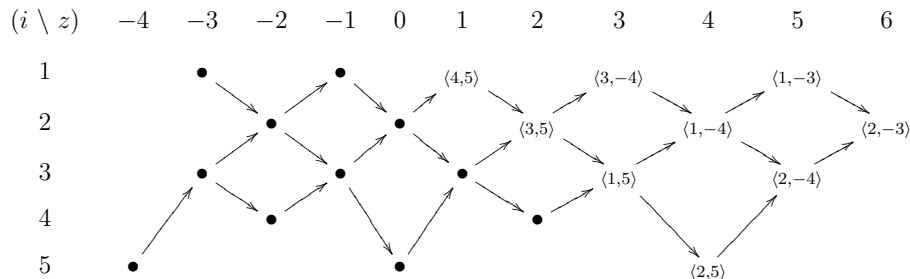
(See the labeled vertices in (4.4).) We denote by $\Upsilon_{[\mathbf{i}_0]}^R$ the full subquiver consisting of the vertices in (i) and (ii) in (4.3)

Corollary 4.13. *Let $\mathbf{i}_0, \mathbf{j}_0, Q$ and t be in Proposition 4.11 and $j_{t_1} = n$ (resp. $j_{t_1} = n+1$). The subquiver $\Upsilon_{[\mathbf{i}_0]}^R$ of twisted AR-quiver $\Upsilon_{[\mathbf{i}_0]}$ is isomorphic to a subquiver Γ_Q^R of the AR-quiver Γ_Q . Moreover, labels in $\Upsilon_{[\mathbf{i}_0]}^R$ direct follow from those in Γ_Q^R . Precisely, correspondences between the labels of Γ_Q^R and $\Upsilon_{[\mathbf{i}_0]}^R$ are as follows:*

$$[i, j] \mapsto \begin{cases} \langle i, -j-1 \rangle & \text{if } i, j \neq n \\ \langle i, -n-1 \rangle \text{ (resp. } \langle i, n+1 \rangle) & \text{if } j = n. \end{cases}$$

Proof. If $j_{t_1} = n$, our assertion directly follows from Proposition 4.11. If $j_{t_1} = n+1$, our assertion can be obtained by swapping the roles of α_n and α_{n+1} . \square

Example 4.14. Let $\mathbf{i}_0 = \prod_{k=0}^4 (2 \ 1 \ 3 \ 5)^{k\vee}$. Then we can find labels using Corollary 4.13 as follows:



The labels of ${}_1\Gamma_Q$ can be obtained from Γ_Q where

$$(4.4) \quad Q = \circ_1 \longrightarrow \circ_2 \longleftarrow \circ_3 \longleftarrow \circ_4 \quad \text{and} \quad \Gamma_Q =$$

$(i \setminus z) \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$

(Here the path consisting of $\{[4], [3, 4], [1, 4], [2, 4]\}$ is the ρ in (4.3).) Hence

$$Q^* = \circ_1 \longrightarrow \circ_2 \longrightarrow \circ_3 \longleftarrow \circ_4 \quad \text{and} \quad \Gamma_{Q^*} \simeq {}_2\Gamma_{Q^*} \simeq$$

as quivers.

Remark 4.15. By taking reflection with respect to x -axis to ${}_2\Gamma_{Q^*}$, one can easily check that ${}_2\Gamma_{Q^*}$ is isomorphic to Γ_Q as quivers. Then the S -sectional path (resp. N -sectional path) in ${}_2\Gamma_{Q^*}$ of length k corresponds to the N -sectional path (resp. S -sectional path) in ${}_1\Gamma_Q$ of length k .

Recall that there is an involution $*$ on the index set I_{n+1}^D such that $w_0(\alpha_i) = -\alpha_{i^*}$.

$$(4.5) \quad \text{If } n+1 \text{ is odd then } * : i \mapsto \begin{cases} i & \text{if } i \neq n, n+1, \\ i + (-1)^{\delta_{n+1, i}} & \text{if } i = n, n+1. \end{cases} \quad \text{On the other hand,}$$

if $n+1$ is even then $i^* = i$, for $i \in I$.

For any reduced expression $\mathbf{i}_0 = i_1 i_2 \cdots i_N$ of w_0 of type D_{n+1} ,

$$s_{i_1} s_{i_2} \cdots s_{i_p} = w_0 s_{i_N} s_{i_{N-1}} \cdots s_{i_{p+1}}.$$

Since $s_{i_p}(\alpha_{i_p}) = -\alpha_{i_p}$, we have $s_{i_1} s_{i_2} \cdots s_{i_p}(\alpha_{i_p}) = -\beta_p^{\mathbf{i}_0}$ and

$$-\beta_p^{\mathbf{i}_0} = w_0 s_{i_N} s_{i_{N-1}} \cdots s_{i_{p+1}}(\alpha_p) = -(s_{i_N} s_{i_{N-1}} \cdots s_{i_{p+1}}(\alpha_{i_p}))^*,$$

where $(\alpha + \beta)^* := \alpha^* + \beta^*$ for $\alpha, \beta \in \Phi^+$ and $-w_0(\alpha_i) = (\alpha_i)^* = \alpha_{i^*}$. Hence

$$(4.6) \quad s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}) = (s_{i_N} s_{i_{N-1}} \cdots s_{i_{p+1}}(\alpha_{i_p}))^* = s_{i_N}^* s_{i_{N-1}}^* \cdots s_{i_{p+1}}^*(\alpha_{i_p}^*).$$

Proposition 4.16. Let $[\mathbf{i}_0]$ be a twisted adapted class. For $\mathbf{j}'_0 = j_1 j_2 \cdots j_N \in [\mathbf{i}_0]$, suppose t_1 and t are the largest integer and second largest integers such that $j_{t_1}, j_t \in \bar{n}$. Let $j_{t_1} = n$ (resp. $j_{t_1} = n+1$). Consider the Dynkin quiver Q of A_n satisfying $\mathbf{p}_{A_n}^{D_{n+1}}([\mathbf{i}_0]) = [Q]$ and Q^{rev} , the Dynkin quiver which is obtained by reversing every arrow in Q . Then there is a reduced expression $\mathbf{k}_0 \in [Q^{\text{rev}}]$ of A_n starting with $j_N j_{N-1} \cdots j_{t+1}$ (resp. $(j_N j_{N-1} \cdots j_{t+1})^\vee$).

Proof. Consider $[\mathbf{i}'_0]$ such that $j_N j_{N-1} \cdots j_1 \in [\mathbf{i}'_0]$. Note the following facts:

- If $[i_1 i_2 \cdots i_n]^\vee$ is a twisted Coxeter element then so is $[i_n i_{n-1} \cdots i_1]^\vee$.
- If $\mathbf{p}_{A_n}^{D_{n+1}}([i_1 i_2 \cdots i_n]^\vee) = \phi_Q$, then $\mathbf{p}_{A_n}^{D_{n+1}}([i_n i_{n-1} \cdots i_1]^\vee) = \phi_{Q^{\text{rev}}}$.

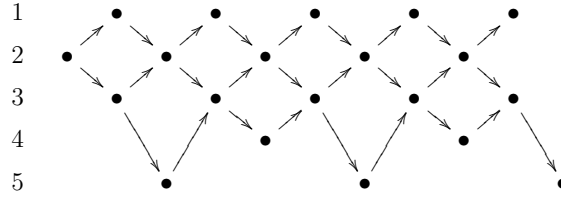
Thus, by Theorem 3.10, we have $[\mathbf{i}'_0]$ is twisted adapted and $\mathbf{p}_{A_n}^{D_{n+1}}([\mathbf{i}'_0]) = [Q^{\text{rev}}]$. By the same argument as Proposition 4.11, we can prove the proposition. \square

Remark 4.17. In the above proposition, we can assume that \mathbf{j}'_0 is a reduced expression in $[\mathbf{i}_0]$ with the smallest t without any loss of generality as in Remark 4.12. By Theorem 2.24 and Theorem 4.3, such \mathbf{j}_0 can be obtained by the following way:

- (4.7) (i) Read the residues of vertices except the residues of all vertices in the left part of length $n-1$ N -sectional path ρ' inside ${}_2\Gamma_{Q^*}$ (ii) Read vertices in ρ' , (iii) Read the remained.

(See the vertices $\{\langle 1, -2 \rangle, \langle 1, -5 \rangle, \langle 2, -5 \rangle, \langle 3, -5 \rangle, \langle 4, -5 \rangle\}$ in Example 4.24.) We denote by $\Upsilon_{[\mathbf{i}_0]}^L$ the full subquiver consisting of the vertices described in (i) and (ii) of (4.7)

Example 4.18. For $\mathbf{i}_0 = \prod_{k=0}^4 (2 \ 1 \ 3 \ 5)^{k^\vee} \in [\mathbf{i}_0]$ in Example 4.14, $\Upsilon_{[\mathbf{i}_0]}^L$ for $\mathbf{i}'_0 = \prod_{k=0}^4 (5 \ 3 \ 1 \ 2)^{k^\vee}$ can be described as follows:



where

$$(4.8) \quad Q^{\text{rev}} = \circ_1 \xleftarrow{\quad} \circ_2 \xrightarrow{\quad} \circ_3 \xrightarrow{\quad} \circ_4 \quad \text{and} \quad \Gamma_{Q^{\text{rev}}} \quad \begin{array}{c} (i \setminus z) \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Corollary 4.19. Let \mathbf{i}_0 , \mathbf{j}_0 , Q and t be in Proposition 4.16 and $j_{t_1} = n^*$. The subquiver $\Upsilon_{[\mathbf{i}_0]}^L$ of twisted AR-quiver $\Upsilon_{[\mathbf{i}_0]}$ is isomorphic to a subquiver $\Gamma_{Q^{\text{rev}}}^L$ inside of $\Gamma_{Q^{\text{rev}}}$. Moreover, labels in $\Upsilon_{[\mathbf{i}_0]}^L$ directly follow from those in $\Gamma_{Q^{\text{rev}}}^L$ in the same way of Corollary 4.13.

Proof. It directly follows from Proposition 4.16. \square

The following lemma is easy to check, we leave the proof for readers.

Lemma 4.20. Let $\mathbf{i}_0^A = i_1 i_2 \cdots i_{n-1} i_n$ be a reduced expression of type A_n .

- (1) We have $\beta_p^{\mathbf{i}_0^A} = s_{i_n}^* s_{i_{n-1}}^* \cdots s_{i_{p+1}}^* (\alpha_{i_p})$.
- (2) If \mathbf{i}_0^A is adapted to Q with the Coxeter element $\phi_Q = s_{i'_1} s_{i'_2} \cdots s_{i'_{n-1}} s_{i'_n}$ then $(\mathbf{i}_0^A)^{-1} := i_n i_{n-1} \cdots i_2 i_1$ is adapted to $Q^{\text{rev}*}$ with the Coxeter element $\phi_{Q^{\text{rev}*}} = s_{i_n^*} s_{i_{n-1}^*} \cdots s_{i_2^*} s_{i_1^*}$.
- (3) If \mathbf{i}_0^A is adapted to the Dynkin quiver Q then $(\mathbf{i}_0^A)^{-1*} = i_n^* i_{n-1}^* \cdots i_2^* i_1^*$, for $*$: $I_n \rightarrow I_n$ such that $i \mapsto n+1-i$, adapted to the Dynkin quiver Q^{rev} .

- (4) Let \mathbf{i}_0 , \mathbf{j}_0 , Q and t be in Proposition 4.16 and let $j_{t_1} = n$. There is an adapted reduced expression \mathbf{j}_0^A to Q which ends with $j_{t+1}^* j_{t+2}^* \cdots j_{l-1}^* j_l^*$, where $*$: $I_n \rightarrow I_n$, $i \mapsto n+1-i$.

Corollary 4.21. Let Q be in Proposition 4.16. We obtain $\Gamma_{Q^{\text{rev}}}$ from Γ_Q by reversing all the arrow in Γ_Q . Hence $\Gamma_{Q^{\text{rev}}}^{\text{rev}}$ is isomorphic to Γ_Q via the map $\alpha \mapsto \alpha$ for $\alpha \in \Phi^+$. Also, if α is in the i -th residue in Γ_Q then α is in the $(n+1-i)$ -th residue in $\Gamma_{Q^{\text{rev}}}$.

Proof. It follows from (1) and (3) in Lemma 4.20. \square

Corollary 4.22. Let t_1 and Q be in Proposition 4.16. We obtain labels of $\Upsilon_{[\mathbf{i}_0]}^L$ from the labels of Γ_Q^L . Precisely, correspondences between the labels of Γ_Q^L and $\Upsilon_{[\mathbf{i}_0]}^L$ are as follows: If $j_{t_1} = n^*$, we have

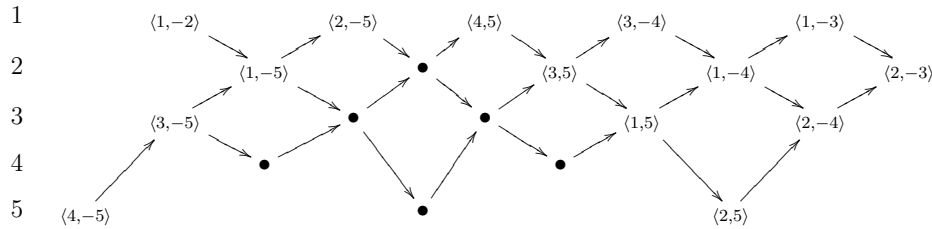
$$[i, j] \mapsto \begin{cases} \langle i, -j-1 \rangle & \text{if } i, j \neq n \\ \langle i, -n-1 \rangle \text{ (resp. } \langle i, n+1 \rangle) & \text{if } j = n. \end{cases}$$

If $j_{t_1} = (n+1)^*$ then we get the subquiver $\Gamma_{[Q]}^L$ by substituting every α_{n+1} in the subquiver $\Upsilon_{[\mathbf{i}_0]}^L$ with α_n .

Proof. It directly follows from Lemma 4.20. \square

Remark 4.23. In Corollary 4.22, the assumption $j_{t_1} = n^*$ can be replaced by the twisted Coxeter element $\phi_{[\mathbf{i}_0]}$ contains $n+1$. On the other hand, the assumption $j_{t_1} = (n+1)^*$ can be replaced by the twisted Coxeter element $\phi_{[\mathbf{i}_0]}$ contains n .

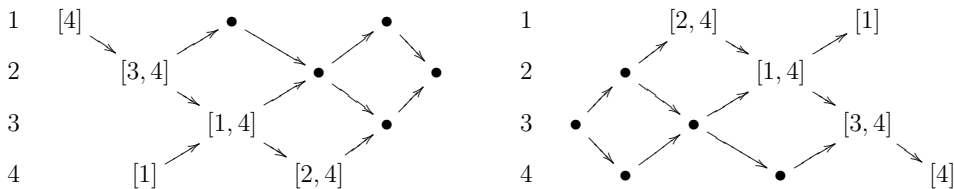
Example 4.24. Let $\mathbf{i}_0 = \prod_{k=0}^4 (2 \ 1 \ 3 \ 5)^{k\vee}$. Then we can add labels to Example 4.14 using Corollary 4.19 and Corollary 4.22 as follows:



The left part of $\Upsilon_{[\mathbf{i}_0]}$ can be obtained from Γ_Q or $\Gamma_{Q^{\text{rev}}}$ where

$$Q = \circ \xrightarrow{1} \circ \xleftarrow{2} \circ \xleftarrow{3} \circ_4$$

and AR-quivers Γ_Q and $\Gamma_{Q^{\text{rev}}}$ are



Convention 4.25. Now, we fix the height function ξ in Algorithm 4.6 such that

$$(4.9) \quad \xi(\bar{1}) - 2 \times a_1^Q + 1 = 0.$$

Note that the ρ (resp. ρ') in (4.3) (resp. (4.7)) is the rightmost (resp. leftmost) S -sectional (resp. N -sectional) path of length $n - 1$ in $\Upsilon_{[i_0]}$. Furthermore, by [20, Remark 1.14], (4.9) implies that

$$\text{vertices in } \rho' \text{ are } (1, -1) \text{ and } \rho \text{ starts at } (1, 1).$$

Equivalently, we can say

$$\text{in } \hat{\Upsilon}_{[i_0]}, \text{ vertices in } \rho' \text{ are } (\bar{r}, -r) \text{ and vertices } \rho \text{ are } (\bar{r}, r).$$

Now, with the above convention, we can summarize the previous results on the labeling $\Upsilon_{[i_0]}$ in this subsection as follows:

Proposition 4.26. *For a twisted adapted class $[\mathbf{i}_0]$, we can label the vertices whose coordinates in $\hat{\Upsilon}_{[i_0]}$ are*

$$\{(\bar{i}, j), (\bar{i}, k) \mid 1 \leq i \leq n, j \leq -i, k \geq i\}$$

by using the labels of Γ_Q . In particular, the set of coordinates of

$$(4.10) \quad \Phi^+|_{n+1} := \{\langle i, n+1 \rangle, \langle i, -n-1 \rangle \mid 1 \leq i \leq n\}$$

in $\hat{\Upsilon}_{[i_0]}$ is $\{(\bar{i}, -i), (\bar{i}, i) \mid 1 \leq i \leq n\}$.

Proof. Note the following:

$$(4.11) \quad \text{In } \Gamma_Q, \rho \text{ in Remark 4.12 and } \rho' \text{ in Remark 4.17 correspond to } S\text{-sectional path of length } n-1 \text{ whose second component is } n, \text{ simultaneously.}$$

Then our assertion follows from Corollary 4.13 and Corollary 4.19. \square

Corollary 4.27. *If the coordinate in $\hat{\Upsilon}_{[i_0]}$ of $\langle j, n+1 \rangle$ is (\bar{i}, i) then the coordinate of $\langle j, -n-1 \rangle$ is $(\overline{n+1-i}, -n-1+i)$. On the other hand, if the coordinate in $\hat{\Upsilon}_{[i_0]}$ of $\langle j, n+1 \rangle$ is $(\bar{i}, -i)$ then the coordinate of $\langle j, -n-1 \rangle$ is $(\overline{n+1-i}, n+1-i)$.*

Proof. It follows from the fact that if the vertices with coordinates (i, i) , $1 \leq i \leq n$, are determined by the Γ_Q then the vertices with coordinates $(i, -i)$ $1 \leq i \leq n$, are also obtained by the map (vertices in i -th residue) \mapsto (vertices in $n+1-i$ -th residue) on Γ_Q . \square

Definition 4.28. For any $\gamma \in \Phi^+$, the *folded multiplicity* of γ , denoted by $\overline{\mathbf{m}}(\gamma)$ is a positive integer defined as follows:

$$\overline{\mathbf{m}}(\gamma) = \max \left\{ \sum_{j \in \bar{i}} m_j \mid \bar{i} \in \bar{I} \text{ and } \sum_{i \in I} m_i \alpha_i = \gamma \right\}.$$

If $\overline{\mathbf{m}}(\gamma) = 1$, we say that γ is *folded multiplicity free*.

Remark 4.29. Note that every folded multiplicity free positive root is multiplicity free. However, $\gamma = \langle a, n \rangle$ of $\Phi_{D_{n+1}}^+$ has $\mathbf{m}(\gamma) = 1$ and $\overline{\mathbf{m}}(\gamma) = 2$.

From now on, we shall label the vertices which are not covered by Proposition 4.26. By the observation (4.11), the number of such vertices is

$$\begin{aligned} \frac{n(n-1)}{2} &= n(n+1) - \frac{n(n+1)}{2} - n \\ &= |\Phi_{D_{n+1}}^+| - |\Phi_{A_n}^+| - \text{the number of vertices in } \rho \\ &= |\{\gamma \in \Phi_{D_{n+1}}^+ \mid \overline{m}(\gamma) = 2\}|. \end{aligned}$$

Furthermore, the vertices satisfy the following condition: Let us denote by $\Upsilon_{[i_0]}^C$ the full subquiver consisting of the uncovered vertices. Then

- each vertex in $\Upsilon_{[i_0]}^C$ is contained in two sectional paths of length $n-1$,
 - $\Upsilon_{[i_0]}^C$ is surrounded by ρ in Remark 4.12 and ρ' in Remark 4.17.
- (4.12) Thus, as the set of vertices, we have

$$\Upsilon_{[i_0]} = \Upsilon_{[i_0]}^L \sqcup \Upsilon_{[i_0]}^C \sqcup \Upsilon_{[i_0]}^R.$$

Proposition 4.30. *Let β_1 and β_2 be positive roots of $\Phi^+_{|_{n+1}}$ such that whose twisted coordinates are $(i_1, -i_1)$ and (i_2, i_2) for $1 \leq i_1, i_2 \leq n-1$. If $i_1 + i_2 \leq n$ then the label with the coordinate $(\overline{i_1 + i_2}, -i_1 + i_2)$ in $\widehat{\Upsilon}_{[i_0]}$ is $\beta_1 + \beta_2$.*

Proof. Suppose the vertex whose coordinate is $(\overline{i_1 + i_2}, -i_1 + i_2)$ in $\widehat{\Upsilon}_{[i_0]}$ has a residue less than n . In order to see the case when the third coordinate of $(\overline{i_1 + i_2}, -i_1 + i_2)$ is 1, let V be the set of vertices which can be described as follows: $V = \cup_{k=0}^{i_1} V_t$, where

$$\begin{aligned} V_0 &= \{(i_2, i_2), (i_2 - 1, i_2 - 1), \dots, (1, 1)\}, \\ V_t &= \{(i_2 + t, i_2 - t), (i_2 + t - 1, i_2 - t - 1), \dots, (t, -t)\}, \text{ for } t = 1, \dots, i_1. \end{aligned}$$

Hence there is a compatible reading of $\Upsilon_{[i_0]}$, equivalently the Q -adapted reduced word of w_0 , which is the form of $\mathbf{j} J_1^{i_2} J_1^{i_2+1} J_1^{i_2+2} \dots J_{i_1}^{i_1+i_2} \mathbf{j}'$ where

- $J_i^j = j j - 1 j - 2 \dots i + 1 i$,
- \mathbf{j}, \mathbf{j}' are some Q -adapted reduced words of Weyl group of type D_{n+1} ,

by reading compatibly (i) the vertices right to V first, (ii) the vertices in V second and (iii) the vertices left to V last. Denote

$$S_{\mathbf{j}} := s_{j_1} s_{j_2} \dots s_{j_k} \text{ for } \mathbf{j} = j_1 j_2 \dots j_k \quad \text{and} \quad S_{t_1}^{t_2} := S_{j_{t_1}^{t_2}} \text{ for } 1 \leq t_1 < t_2 \leq n.$$

Then the label with coordinate (i_2, i_2) is $S_{\mathbf{j}}(\alpha_{i_2})$. The label with coordinate $(i_1, -i_1)$ is

$$S_{\mathbf{j}} s_1^{i_2} s_1^{i_2+1} s_2^{i_2+2} \dots s_{i_1-1}^{i_1+i_2-1} s_{i_1+1}^{i_1+i_2}(\alpha_{i_1}) = S_{\mathbf{j}} \left(\sum_{k=i_2+1}^{i_1+i_2} \alpha_k \right) \in \Phi^+,$$

since one can easily compute that $S_1^{i_2} S_1^{i_2+1} S_2^{i_2+2} \dots S_{i_1-1}^{i_1+i_2-1} S_{i_1+1}^{i_1+i_2}(\alpha_{i_1}) = \sum_{k=i_2+1}^{i_1+i_2} \alpha_k \in \Phi^+$. Thus the label with coordinate $(i_1 + i_2, -i_1 + i_2)$ is

$$S_j S_1^{i_2} S_1^{i_2+1} S_2^{i_2+2} \dots S_{i_1-1}^{i_1+i_2-1}(\alpha_{i_1+i_2}) = S_j \left(\sum_{k=i_2}^{i_1+i_2} \alpha_k \right) \in \Phi^+.$$

Since $S_j(\alpha_{i_2}) + S_j \left(\sum_{k=i_2+1}^{i_1+i_2} \alpha_k \right) = S_j \left(\sum_{k=i_2}^{i_1+i_2} \alpha_k \right)$, we proved the assertion.

Also, when the third coordinate of $(\overline{i_1 + i_2}, -i_1 + i_2)$ is -1 , our assertion can be proved by the same argument. \square

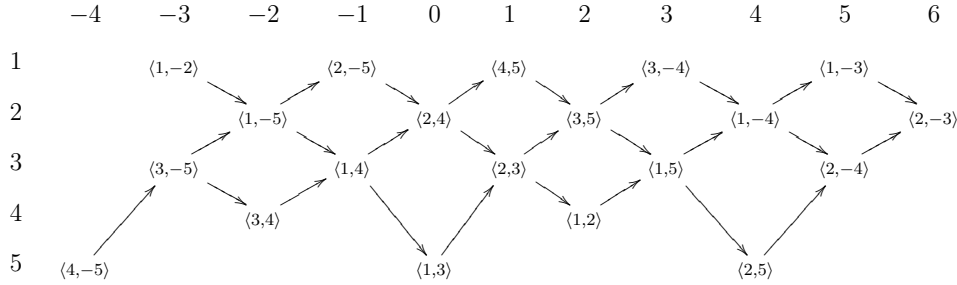
By (4.12), we have the following proposition.

Proposition 4.31. *For each vertex v in $\Upsilon_{[i_0]}^C$, there exist unique $\alpha \in \rho$ and $\beta \in \rho'$ such that*

- *whose folded coordinates are $(\overline{i_1}, i_1)$ and $(\overline{i_2}, -i_2)$ respectively,*
- *$(\overline{i_1 + i_2}, -i_1 + i_2)$ coincides with the folded coordinate of v .*

Proof. By taking α as the intersection of ρ and the N -sectional path containing v and β as the intersection of ρ' and the S -sectional path containing v , our assertion follows. \square

Example 4.32. Let $i_0 = \prod_{k=0}^4 (2 \ 1 \ 3 \ 5)^{k \vee}$. Then we can add labels to Example 4.24 using Proposition 4.30 as follows:



We can see that $\langle 2, 3 \rangle = \langle 2, -5 \rangle + \langle 3, 5 \rangle$, $\langle 1, 3 \rangle = \langle 1, -5 \rangle + \langle 3, 5 \rangle$ and so on.

Let us denote by

$$\nu_{[i_0]} : {}_1\Gamma_Q \sqcup {}_2\Gamma_{Q^*} \rightarrow \Upsilon_{[i_0]},$$

the map between the sets of vertices. Recall Remark 4.15 for the isomorphism of quivers ${}_1\Gamma_Q \simeq {}_2\Gamma_{Q^*} \simeq \Gamma_Q$.

Proposition 4.33. *For a twisted adapted class $[i_0]$, we have*

$$\nu_{[i_0]}^{-1}(\langle i, j \rangle) = \sum_{k=i}^{j-1} \alpha_k = [i, j-1]$$

for $1 \leq i < j \leq n$.

Proof. Suppose the twisted Coxeter element of $[i_0]$ has n and $\nu_{[i_0]}^{-1}(\langle i, j \rangle) \in {}_1\Gamma_Q$. Note that

- (1) $\langle i, j \rangle = \langle i, -n-1 \rangle + \langle j, n+1 \rangle$,
- (2) $\nu_{[i_0]}^{-1}(\langle i, -n-1 \rangle) \in {}_1\Gamma_Q$,

$$(3) \nu_{[i_0']}^{-1}(\langle j, n+1 \rangle) \in {}_2\Gamma_{Q^*}.$$

If we denote the coordinate of $\langle i, -n-1 \rangle$ and $\langle j, n+1 \rangle$ in $\widehat{\Upsilon}_{[i_0]}$ by $(\overline{i_1}, i_1)$ and $(\overline{i_2}, -i_2)$ then the coordinate of $\langle i, j \rangle$ and $\langle j, -n-1 \rangle$ are $(\overline{i_1 + i_2}, i_1 - i_2)$ and $(\overline{n+1 - i_2}, n+1 - i_2)$. Then, by taking proper height function of type A_n , we have

- (a) coordinates of $[i, n]$ and $[j, n]$ in Γ_Q are (i_1, i_1) and $(n+1 - i_2, n+1 - i_2)$,
- (b) the root α with the coordinate $(i_1 + i_2, i_1 - i_2)$ in Γ_Q satisfies that $\alpha + [j, n] = [i, n]$, i.e., $\alpha = [i, j-1]$.

Here (b) follows by the additive property of AR-quiver. For the remaining cases. we can apply the similar arguments. \square

By summing up results in this section, we get the following algorithm for labeling twisted and folded AR-quiver of type D_{n+1} by using the labels of Γ_Q of type A_n .

Algorithm 4.34. Let Q be the Dynkin quiver of type A_n such that $\mathfrak{p}_{A_n}^{D_{n+1}}([i_0]) = [Q]$

- (i) Let us consider take ${}_1\Gamma_Q$ as Γ_Q with the labeling $\Phi_{A_n}^+$ and take ${}_2\Gamma_{Q^*}$ by turning ${}_1\Gamma_Q$ upside down.
- (ii) Put ${}_2\Gamma_{Q^*}$ at the left of ${}_1\Gamma_Q$ in a canonical way.
- (iii) Take ρ the unique S -sectional path of length $n-1$ in ${}_1\Gamma_Q$ and take ρ' the unique N -sectional path of length $n-1$ in ${}_2\Gamma_{Q^*}$. Then their labels are the same as $\{[i, n] \mid 1 \leq i \leq n\}$
- (iv) Change all labels right to ρ or left to ρ' from $[a, b]$ to $\langle a, -b-1 \rangle$.
- (v) Change the duplicated labels $\{[i, n] \mid 1 \leq i \leq n\}$ in $\rho \sqcup \rho'$ as follows:

$$\begin{aligned} {}_1\Gamma_Q \ni [i, n] &\mapsto \begin{cases} \langle i, -n-1 \rangle & \text{if } n \text{ appears in } \phi_{[i_0]}, \\ \langle i, n+1 \rangle & \text{if } n+1 \text{ appears in } \phi_{[i_0]}, \end{cases} \\ {}_2\Gamma_{Q^*} \ni [i, n] &\mapsto \begin{cases} \langle i, -n-1 \rangle & \text{if } n+1 \text{ appears in } \phi_{[i_0]}, \\ \langle i, n+1 \rangle & \text{if } n \text{ appears in } \phi_{[i_0]}. \end{cases} \end{aligned}$$

- (vi) Change all labels surrounded by ρ and ρ' from $[a, b]$ to $\langle a, b+1 \rangle$.

The resulting quiver is the folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$.

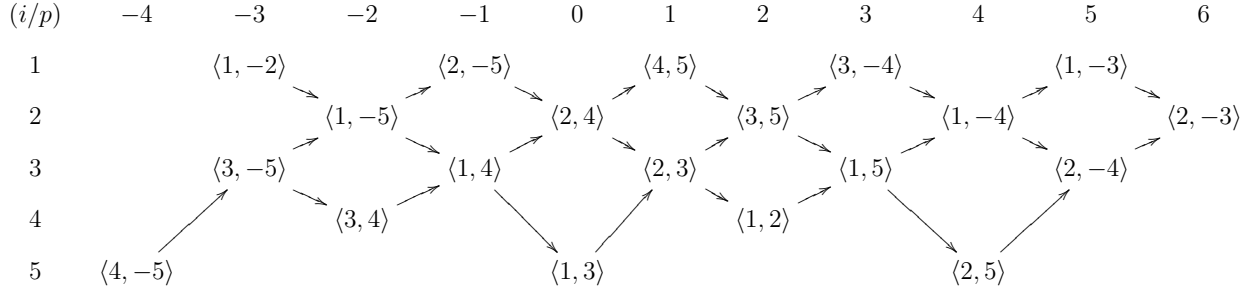
Remark 4.35. From the above algorithm, one can notice that

- (i) for any $\beta \in \Upsilon_{[i_0]}^L$ and $\alpha \in \Upsilon_{[i_0]}^R$, there exists a path from β to α inside $\Upsilon_{[i_0]}$,
- (ii) there exists a path from β to α in Γ_Q if and only if there exists a path from β' to α' in ${}_1\Gamma_Q$ (resp. ${}_2\Gamma_{Q^*}$), where β' and α' correspond to labels β and α obtained from the algorithm.

From Algorithm 4.34, we have also an interesting phenomena as follows:

Corollary 4.36. Assume that we have $\gamma \in \Phi^+ \setminus \Pi$. Then γ is folded multiplicity non-free if and only if γ is contained in $\Upsilon_{[i_0]}^C$.

Finally, we get the twisted AR-quiver:



Remark 4.39. We have another algorithm to get labels of twisted AR-quivers without using the labels of Γ_Q of type A_n . See Algorithm 5.12.

5. LABELING OF A TWISTED AR-QUIVER USING ONLY ITS SHAPE

As in the last section, we fix the height function of a folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$ by letting $\langle i, \pm n + 1 \rangle$ have the coordinate $(\bar{j}, \pm j)$. Note that we can naturally define the notion of the sectional path for twisted AR-quivers and folded AR-quivers of type D_n . Now we shall extend the notion of the sectional path for our purpose:

Definition 5.1. Let us fix $m \in \mathbb{Z}$.

- (1) An N_m^{ext} -sectional quiver is the subquiver of $\widehat{\Upsilon}_{[i_0]}$ consisting of the following set of vertices $\{(\bar{i}, -i + 2m + 2(n + 1)m') \mid 0 \leq i \leq n, m' \in \mathbb{Z}\} \cap \widehat{\Upsilon}_{[i_0]}$.
- (2) An S_m^{ext} -sectional quiver is the subquiver of $\widehat{\Upsilon}_{[i_0]}$ consisting of the following set of vertices $\{(\bar{i}, i + 2m + 2(n + 1)m') \mid 0 \leq i \leq n, m' \in \mathbb{Z}\} \cap \widehat{\Upsilon}_{[i_0]}$.
- (3) An N_m -sectional path is the subquiver of $\widehat{\Upsilon}_{[i_0]}$ consisting of the following set of vertices $\{(\bar{i}, -i + 2m) \mid 0 \leq i \leq n\} \cap \widehat{\Upsilon}_{[i_0]}$.
- (4) An S_m -sectional path is the subquiver of $\widehat{\Upsilon}_{[i_0]}$ consisting of the following set of vertices $\{(\bar{i}, i + 2m - (2n + 2)) \mid 0 \leq i \leq n\} \cap \widehat{\Upsilon}_{[i_0]}$.
- (5) An m -swing is the union of the N_m -sectional path and the S_m -sectional path.
- (6) An m -snake is the union of the N_m^{ext} -sectional path and the S_m^{ext} -sectional path associated to the same m .

Example 5.2. In Example 4.38, we can observe properties of the notions in Definition 5.1:

- (a) N_{-1}^{ext} -sectional quiver is union of N_{-1} -sectional path and N_4 -sectional path which are disconnected. Here, N_{-1} -sectional path consists of $\{\langle 1, -2 \rangle\}$ and N_4 -sectional path consists of $\{\langle 2, -3 \rangle, \langle 2, -4 \rangle, \langle 2, 5 \rangle\}$. On the other hand, the N_0^{ext} -sectional quiver is connected and coincides with ρ' consisting of $\{\langle 2, -5 \rangle, \langle 1, -5 \rangle, \langle 3, -5 \rangle, \langle 4, -5 \rangle\}$.
- (b) S_2^{ext} -sectional quiver is union of S_2 -sectional path and S_{-3} -sectional path which are disconnected. Here, S_2 -sectional path consists of $\{\langle 1, -3 \rangle, \langle 2, -3 \rangle\}$ and S_{-3} -sectional path consists of $\{\langle 3, -5 \rangle, \langle 3, 4 \rangle\}$. On the other hand, the S_{-1}^{ext} -sectional quiver is connected and coincides with S_{-1} -sectional path consisting of $\{\langle 2, -5 \rangle, \langle 2, 4 \rangle, \langle 2, 3 \rangle, \langle 1, 2 \rangle\}$.

The sectional paths have the following propositions.

Proposition 5.3.

- (1) Each N_m^{ext} -sectional (resp. S_m^{ext} -sectional) quiver consists of n vertices with distinct residues $\bar{1}, \bar{2}, \dots, \bar{n}$.
- (2) If $m_1 \equiv m_2 \pmod{n+1}$ then $N_{m_1}^{\text{ext}}$ -sectional quiver (resp. $S_{m_1}^{\text{ext}}$ -sectional quiver) coincides with $N_{m_2}^{\text{ext}}$ -sectional quiver (resp. $S_{m_2}^{\text{ext}}$ -sectional quiver).
- (3) If $m_1 \not\equiv m_2 \pmod{n+1}$ the $N_{m_1}^{\text{ext}}$ -sectional quiver and the $S_{m_2}^{\text{ext}}$ -sectional quiver have a unique intersection.
- (4) For each m , one of the N_m^{ext} -sectional quiver and S_m^{ext} -sectional quiver is connected.
- (5) If $\langle k_1, k_2 \rangle$ is in the N_m -sectional (resp. S_m -sectional) path then $\langle k_1, -k_2 \rangle$ is in the S_m^{ext} -sectional (resp. N_m^{ext} -sectional) quiver.

Proof. (1), (2) and (3) are not hard to prove so that we only give a proof of (4) and (5). By the observations in the previous section, we know that $\langle 1, n+1 \rangle$ and $\langle 1, -n-1 \rangle$ have coordinates $(\bar{i}, \pm i)$ and $(\bar{j}, \pm j)$ with $i+j = n+1$. Recall the facts in [20, Corollary 1.15]:

- (a) In an AR-quiver Γ_Q of type A_n , $[1, n]$ is located at the intersection of the S -sectional path of length $n-1$ and the N -sectional path of length $n-1$.
- (b) In an AR-quiver Γ_Q of type A_n , there are exactly two sectional path of length $n-1$.

Since folded AR-quiver $\hat{\Upsilon}_{[i_0]}$ of D_{n+1} can be understood as the disjoint union of AR-quivers ${}_1\Gamma_Q$ and ${}_2\Gamma_{Q^*}$ of type A_n , we conclude that every N_m^{ext} -sectional path associated to $m = 0, 1, \dots, i$ and every S_m^{ext} -sectional path associated to $m = 0, 1, \dots, j$ is connected. Then our fourth assertion follows from the fact that $i+j = n+1$.

For (5), recall that (\bar{i}, j) and $(\overline{n+1-i}, n+1+j)$ are in the N_m^{ext} -sectional (resp. S_m^{ext} -sectional) quiver and S_m^{ext} -sectional (resp. N_m^{ext} -sectional) quiver, respectively, for $m = (i+j)/2$ (resp. $m = (j-i)/2$). By Corollary 4.37, we proved the corollary. \square

Proposition 5.4. For each $m \in \mathbb{Z}$, the m -snake has a positive integer p such that every vertex in m -snake has a summand p or $-p$.

Proof. Let us take a connected N_m^{ext} -sectional quiver associated $m \not\equiv 0 \pmod{n+1}$. Then the N_m^{ext} -sectional quiver has an intersection with the S_0^{ext} -sectional. Suppose (\bar{i}, i) is the vertex with the label $\langle j, \pm(n+1) \rangle$. Recall the two following facts:

- (a) vertices in an N -sectional path of A_n share the first component (Theorem 4.3). Thus $(\bar{i}', 2i-i')$ in ${}_1\Gamma_Q \subset \hat{\Upsilon}_{[i_0]}$ for $i' < i$ share j as the first summand.
- (b) Proposition 4.30 shows that $(\bar{i}', 2i-i')$ for $i' > i$ has j as its summand.

Hence we proved for the N -sectional path N^1 contained in N_m^{ext} -sectional quiver and has an intersection with S_0^{ext} -sectional quiver. Thus if N_m^{ext} -sectional quiver is connected, then all vertices in N_m^{ext} share j as their first summand.

Now we assume that N_m^{ext} -sectional quiver is not connected. We claim that

N_m^{ext} -sectional quiver is not connected (equivalently the length of N^1 is k less than $n-1$), the connected component N^1 is contained in ${}_1\Gamma_Q$.

If N_1 is not contained in ${}_1\Gamma_Q$, then it contains a vertex in $\Upsilon_{[i_0]}^C$ which yields a contradiction to the first observation in (4.12). By Theorem 4.3 and Algorithm 4.34, the summand j is

the same as $n - k$, in this case. By (1) in Proposition 5.3, the other N -sectional path N^2 of N_m^{ext} -sectional quiver is of length $n - 2 - k$. By Theorem 4.3, Remark 4.15 and Algorithm 4.34, every vertex in N^2 has $-(n + 1 - k)$ as its second summand and N^2 is contained in ${}_2\Gamma_{Q^*}$. Hence we proved our assertion for N_m^{ext} -sectional quiver. As a summary,

- (i) If N_m^{ext} -sectional quiver is connected, every vertex in it has p as its summand,
- (ii) If N_m^{ext} -sectional quiver is not connected, N_m^{ext} -sectional quiver is decomposed into two connected N -sectional paths, N_m^1 whose vertices share p as their first summand and N_m^2 whose vertices share $-p$ as their second summand.

The assertion restricted to S_m^{ext} -sectional quiver follows from (5) in Proposition 5.3. More precisely, for $m \neq 0$,

- (i) If S_m^{ext} -sectional quiver is connected, then N_m^{ext} -sectional quiver is not connected and every vertex in it has p as its summand,
- (ii) If S_m^{ext} -sectional quiver is not connected, then N_m^{ext} -sectional quiver is connected and S_m^{ext} -sectional quiver is decomposed into two connected N -sectional paths, S_m^1 whose vertices share $-p$ as their second summand and S_m^2 whose vertices share p as their first summand.

□

From the above proposition, we have the following corollaries:

Corollary 5.5.

- (1) If N_m^{ext} -sectional quiver is not connected, it consists of an N -sectional subquiver N_m^1 contained in ${}_1\Gamma_Q$, and an N -sectional subquiver N_m^2 contained in ${}_2\Gamma_{Q^*}$. Furthermore,
 - (N-i) every vertex in N_m^1 has $n - k$ as its first summand and every vertex in N_m^2 has $-(n - k)$ as its second summand, when the length of N_m^1 is of length $k < n - 1$,
 - (N-ii) every vertex in N_m^{ext} is folded multiplicity free.
- (2) If S_m^{ext} -sectional quiver is not connected, it consists of an S -sectional subquiver S_m^1 contained in ${}_1\Gamma_Q$, and an S -sectional subquiver S_m^2 contained in ${}_2\Gamma_{Q^*}$. Furthermore,
 - (S-i) every vertex in S_m^1 has $-(k + 2)$ as its second summand and every vertex in S_m^2 has $k + 2$ as its first summand, when the length of S_m^1 is of length $k < n - 1$.
 - (S-ii) every vertex in S_m^{ext} is folded multiplicity free.

Corollary 5.6. Recall that a_1^Q is the number of arrows in Q directed toward 1. Only when $m = 0$ or $m = \mathbf{a} := a_1^Q + 1$, both N_m^{ext} -sectional quiver and S_m^{ext} -sectional quiver are connected. Moreover,

- (1) every vertex in N_0^{ext} -sectional quiver or S_0^{ext} -sectional quiver is folded multiplicity free and has $n + 1$ or $-n - 1$ as its second summand.
- (2) every vertex in $N_{\mathbf{a}}^{\text{ext}}$ -sectional quiver and $S_{\mathbf{a}}^{\text{ext}}$ -sectional quiver has 1 as its first summand.
- (3) when $m \neq 0$ or \mathbf{a} , one of N_m^{ext} -sectional quiver and S_m^{ext} -sectional quiver is connected and the other is not.

Proof. When $m = 0$ (resp. $m = a_i^Q$), N_m^{ext} -sectional quiver coincides with the N -sectional path of length $n - 1$ in ${}_1\Gamma_Q$ (resp. ${}_2\Gamma_{Q^*}$) and S_m^{ext} -sectional quiver coincides with the S -sectional path of length $n - 1$ in ${}_2\Gamma_{Q^*}$ (resp. ${}_1\Gamma_Q$).

(1) Since every vertex in the N -sectional path of length $n - 1$ in Γ_Q is of the form $[a, n]$, the assertion (1) follows from Remark 4.15 and (iii) in Algorithm 4.34.

(2) Since every vertex in the S -sectional path of length $n - 1$ in Γ_Q is of the form $[1, b]$, the assertion (2) hold by the same reason of (1).

(3) This assertion follows from Theorem 4.3. \square

Now we can rename the notions in Definition 5.1 by considering the above results:

Definition 5.7.

- (1) An m -snake is renamed as a $[p]$ -snake if every vertex of the snake has p or $-p$ as a summand.
- (2) An N_m^{ext} -sectional (resp. S_m^{ext} -sectional) quiver is renamed as the $N^{\text{ext}}[\pm p]$ -sectional (resp. $S^{\text{ext}}[\pm p]$ -sectional) quiver if every vertex of it has p or $-p$ as a summand.
- (3) An X_m -sectional path is renamed as the $X[p]$ -sectional (resp. $X[-p]$ -sectional) path if every vertex of it has p (resp. $-p$) as a summand. Here $X = N$ or S .
- (4) An m -swing is renamed as the $[p]$ -swing (resp. $[-p]$ -swing) if it is the union of the $N[p]$ -sectional path (resp. $N[-p]$ -sectional path) and the $S[p]$ -sectional path (resp. $S[-p]$ -sectional path)

We sometimes call the $N^{\text{ext}}[\pm p]$ -sectional quiver or $S^{\text{ext}}[\pm p]$ -sectional quiver by an extended p -sectional quiver.

Theorem 5.8. *For the folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$, a root $\alpha \in \Phi^+$ has p or $-p$ as a summand if and only if α is in the $[p]$ -snake.*

Proof. Note that the number of roots with p or $-p$ as a summand is the same as $2n$ which is also the number of vertices in the m -snake for any $m \in \mathbb{Z}$. Thus our assertion follows from Proposition 5.4. \square

Corollary 5.9.

- (1) For $p \in \{1, 2, \dots, n+1\}$, a root $\alpha \in \Phi^+$ has p (resp. $-p$) as its summand if and only if α is in the $[p]$ -swing (resp. $[-p]$ -swing).
- (2) The root $\langle a, b \rangle$ is the intersection of the $[a]$ -swing and the $[b]$ -swing.

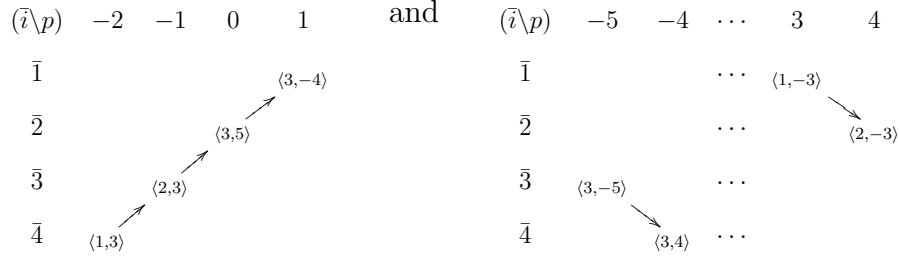
Proof. It is an immediate consequence from the fact that $[p]$ -snake is the union of $[p]$ -swing and $[-p]$ -swing. \square

Corollary 5.10. *Let $\widehat{\Upsilon}_{[i_0]}$ be the folded AR-quiver and let $p \in I \setminus \{n+1\}$.*

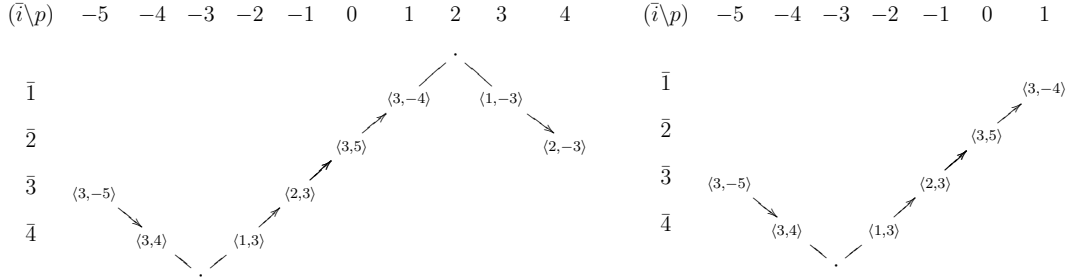
- (1) For any (\bar{n}, k) and $(\bar{n}, k+2)$ in $\widehat{\Upsilon}_{[i_0]}$, these two vertices are in the same swing.
- (2) If a sectional path has a vertex in the \bar{n} -th residue then it is not the $N[-p]$ nor the $S[-p]$ -sectional path, i.e. it is either $N[p]$, $S[p]$, $N[\pm(n+1)]$ or $S[\pm(n+1)]$ -sectional path.

Proof. The first assertion follows from the computation that (\bar{n}, k) and $(\bar{n}, k+2)$ are contained in same m -snake for some m . Note that all vertices with \bar{n} -th residue are contained in $\Upsilon_{[i_0]}^C$, ρ or ρ' . Since ρ (resp. ρ') coincides with S_0^{ext} -sectional quiver (resp. N_0^{ext} -sectional), our second assertion follows from Algorithm 4.34. \square

Example 5.11. In the twisted AR-quiver $\widehat{\Upsilon}_{[i_0]}$ of $[i_0]$ in Example 4.38, the $N[3]$ -sectional path and $S[3]$ -sectional path are



We can see that the $S[3]$ -sectional path is $\langle 3, -5 \rangle \rightarrow \langle 3, 4 \rangle$, the $S[-3]$ -sectional path is $\langle 1, -3 \rangle \rightarrow \langle 2, -3 \rangle$, the $N[3]$ -sectional path is $\langle 1, 3 \rangle \rightarrow \langle 2, 3 \rangle \rightarrow \langle 3, 5 \rangle \rightarrow \langle 3, -4 \rangle$ and $N[-3]$ -sectional path is \emptyset . The following picture is the 3-snake and 3-swing. Also, $[-3]$ -swing is same as $S[-3]$ -sectional path.



Now we can characterize the $[p]$ -snake for $p = 1, 2, \dots, n+1$ as follows:

Let $\{X, Y\} = \{N, S\}$.

- (5.1)
- ($p \neq n+1$ and 1) $[p]$ -snake consists of one X -sectional path of length $n-1$ and two sectional Y -paths Y^1 of length k contained in ${}_1\Gamma_Q$ and Y^2 of length l contained in ${}_2\Gamma_{Q^*}$ such that $k+l = n-2$. If $Y = S$ then $k = p-2$, every vertex in S^1 has $-p$ as its second summand and the remained vertices in $[p]$ -snake have p as their common summand. If $Y = N$, then $k = p-2$, every vertex in N^2 has $-p$ as its second summand and the remained vertices in $[p]$ -snake have p as their common summand.
 - ($p = n+1$) $[n+1]$ -snake consists of an N -sectional path ($= N_0$ -sectional path) and an S -sectional path ($= S_{n+1}$ -sectional path) of length $n-1$ such that the N -sectional path is located at the right of the S -sectional path.
 - ($p = 1$) $[n+1]$ -snake consists of an N -sectional path and an S -sectional path of length $n-1$ such that the S -sectional path is located at the right of the N -sectional path.

Using the properties of sectional quivers (paths), we can find labels of the twisted AR-quiver $\Upsilon_{[i_0]}$ by using only its shape.

Algorithm 5.12.

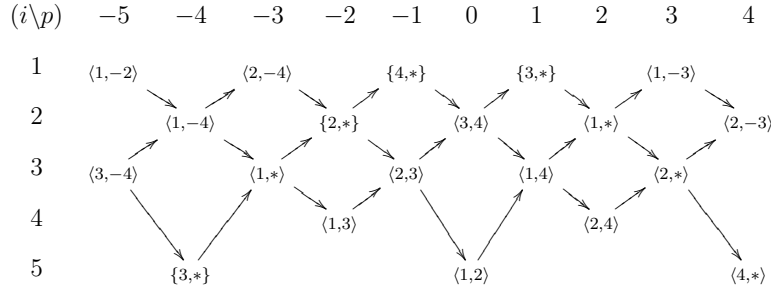
(Step 1) *By using the height function and (2.1), find the shape of $\widehat{\Upsilon}_{[i_0]}$*

(Step 2) *Using the characterizations in (5.1), find all $[p]$ -snake for $p = 1, 2, \dots, n$ and put their summand. In this step, we can complete the labels except those in $[n+1]$ -snake.*

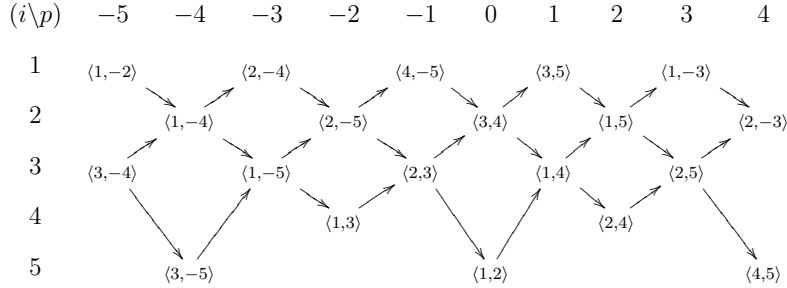
(Step 3) *If n (resp. $n+1$) appears in $\phi_{[i_0]}$, we complete labels for vertices in S_0^{ext} -sectional path by putting $-n-1$ (resp. $n+1$) and labels for vertices in N_0^{ext} -sectional path by putting $n+1$ (resp. $-n-1$).*

Example 5.13. Suppose $[i_0]$ has the twisted Coxeter element $(s_2 s_5 s_1 s_3) \vee$. Here we denote by $\{a, b\}$ the root α with summands a and b .

By (Step 1) and (Step 2) for $p \neq n+1$

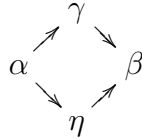


By (Step 3), we can complete the label.



6. TWISTED ADDITIVE PROPERTY OF D_{n+1}

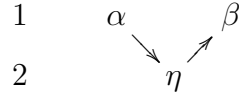
Proposition 6.1. *Let us take a square with length of edge 1 in the folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$:*



- (i) *There is a swing which has both root γ (resp. η) and α . Similarly, there is a swing which has both root γ (resp. η) and β .*
- (ii) *There are four swings which has α or β .*
- (iii) $\alpha + \beta = \eta + \gamma$.

Proof. (i) is obvious. So here we give proofs for (ii) and (iii). Let α be on the $N[\epsilon_1 \cdot a_1]$ and $S[\epsilon_2 \cdot a_2]$ -sectional paths and β be on the $N[\epsilon_1 \cdot b_1]$ and $S[\epsilon_2 \cdot b_2]$ -sectional paths, where $\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2$ are 1 or -1 . It is obvious that $a_1 \neq b_1$ and $a_2 \neq b_2$. Observe that the $N^{\text{ext}}[\pm a_1]$ -sectional (resp. $N^{\text{ext}}[\pm b_1]$ -sectional) quiver and the $S^{\text{ext}}[\pm b_2]$ -sectional (resp. $S^{\text{ext}}[\pm a_2]$ -sectional) quiver have an intersection γ (resp. η). By (3) in Proposition 5.3, we have $a_1 \neq b_2$ and $b_1 \neq a_2$. Hence a_1, b_1, a_2 and b_2 are distinct. Now, by Proposition 5.4 and Corollary 5.5, γ has $\epsilon_1 \cdot a_1$ and $\epsilon_2 \cdot b_2$ as its summands and η has $\epsilon_2 \cdot a_2$ and $\epsilon_1 \cdot b_1$ as its summands, which implies $\alpha + \beta = \gamma + \eta$. \square

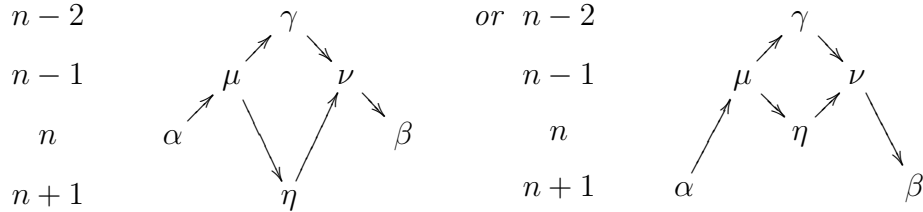
Proposition 6.2. *Let us take three vertices in the first and second residues of twisted AR-quiver $\Upsilon_{[i_0]}$ such that :*



Then we have $\alpha + \beta = \eta$.

Proof. It follows from the additive property of adapted AR-quiver of type A_n if three vertices are induced from ${}_1\Gamma_Q$ (resp. ${}_2\Gamma_{Q^*}$) in Algorithm 4.34. Otherwise, $\alpha = \langle a_1, \pm(n+1) \rangle$ and $\beta = \langle b_1, \pm(-n-1) \rangle$. Then we know $\eta = \alpha + \beta$ by Proposition 4.30. \square

Proposition 6.3. *Let us take six vertices in from $n-2$ -th to $n+1$ -th residues of twisted AR-quiver $\Upsilon_{[i_0]}$ such that :*



Then we have $\alpha + \beta = \eta + \gamma = \mu + \nu$.

Proof. By Algorithm 4.34, we can see that summands of α and β lie in $\{1, 2, \dots, n+1\} \cup \{-n-1\}$. More precisely, if α (resp. β) has negative summand $(-n-1)$ then it is shared by every element in the $N[-n-1]$ (resp. $S[-n-1]$)-sectional path. Let α be on the $N[\epsilon \cdot a_1]$ and $S[a_2]$ -sectional paths and β be on the $N[b_1]$ and $S[\epsilon \cdot b_2]$ -sectional paths. By Corollary 5.10, we know that

- η is on the $N[a_2]$ -sectional and the $S[b_1]$ -sectional paths,
- γ is on the $N^{\text{ext}}[\pm a_1]$ -sectional and $S^{\text{ext}}[\pm b_2]$ -sectional quivers.

Since α and γ (resp. β and γ) are in the same connected component of $N^{\text{ext}}[\pm a_1]$ -sectional (resp. $S^{\text{ext}}[\pm b_2]$ -sectional) quiver, γ is on the $N[\epsilon \cdot a_1]$ and $S[\epsilon \cdot b_2]$ -sectional quivers. Since γ and η have $\epsilon \cdot a_1, a_2, b_1$ and $\epsilon \cdot b_2$ as their summands, we conclude that

$$\alpha + \beta = \eta + \gamma.$$

Now, by Proposition 6.1, our assertion follows. \square

By Proposition 6.1, Proposition 6.2 and Proposition 6.3, we obtain *the twisted additive property* of $\Upsilon_{[i_0]}$, which is stated as follows:

Theorem 6.4 (Twisted additive property of D_{n+1}). *Let α and β be two roots in Φ^+ whose folded coordinates are $(\bar{i}, p - 2^{|\bar{i}|})$ and (\bar{i}, p) in the folded AR-quiver $\hat{\Upsilon}_{[i_0]}$. Here $|\bar{i}|$ denotes the number of indices in the orbit \bar{i} . Then*

$$\alpha + \beta = \sum_{\gamma \in \mathcal{J}} \gamma,$$

where $\mathcal{J} \subset \Phi^+$ consists of γ which are on paths from α to β and have coordinates $(\bar{j}, p - 2^{|\bar{j}| - 1})$ for $\bar{j} \in \bar{I}$.

Remark 6.5. For adapted classes of type ADE_n and twisted adapted classes of type A_{2n+1} , additive properties can be understood as analogous properties of Theorem 6.4.

- (1) In order to see the cases for adapted classes, we can consider ordinary Coxeter elements are associated to the identity map on I . Then the order of every orbit $|\bar{i}| = 1$. (Note that the identity map on I is not compatible with the involution $*$). Observe that, in (2.4), the coordinates of β and $\phi_Q(\beta)$ are (i, p) and $(i, p - 2) = (i, p - 2^{|\bar{i}|})$, respectively. Hence, for $\alpha = \phi_Q(\beta)$, the additive property (2.4) can be rewritten as follows:

$$\alpha + \beta = \sum_{\gamma \in \mathcal{J}} \gamma,$$

where $\mathcal{J} \subset \Phi^+$ consists of γ which are on paths from α to β and have coordinates $(j, p - 2^{|\bar{j}| - 1}) = (j, p - 1)$ for $j \in I$.

- (2) In [24, Proposition 7.7], we proved the analogous statement holds for twisted adapted classes of type A_{2n+1} .

Example 6.6. See the last quiver in Example 4.38. Then the twisted additive property can be checked in this quiver. For example,

$$\langle 2, 4 \rangle + \langle 3, 5 \rangle = \langle 4, 5 \rangle + \langle 2, 3 \rangle, \quad \langle 2, -5 \rangle + \langle 4, 5 \rangle = \langle 2, 4 \rangle, \quad \langle 1, 3 \rangle + \langle 2, 5 \rangle = \langle 3, 5 \rangle + \langle 1, 2 \rangle.$$

7. TWISTED DYNKIN QUIVER AND REFLECTION FUNCTORS

Recall the diagram (2.6). For the twisted adapted classes, we introduced the twisted AR-quivers, the twisted Coxeter elements and the twisted adapted classes. Hence the only thing we want to generalize in (2.6) is a Dynkin quiver Q .

Definition 7.1. A *twisted Dynkin quiver* of D_{n+1} is obtained by giving an orientation to each edge in one of the following diagrams:

$$(7.1) \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ \\ 1 & & 2 & & 3 & & 4 & & n-1 & & \begin{smallmatrix} \bullet \\ \circ \end{smallmatrix} \\ & & & & & & & & & & \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \end{array}$$

$$(7.2) \quad \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ \\ 1 & & 2 & & 3 & & 4 & & n-1 & & \begin{smallmatrix} \otimes \\ \circ \end{smallmatrix} \\ & & & & & & & & & & \begin{smallmatrix} n+1 \\ n \end{smallmatrix} \end{array}$$

We denote by $Q^{\leftarrow n}$ a Dynkin quiver obtained from the diagram (7.1) and by $Q^{\leftarrow n+1}$ a Dynkin quiver obtained from the diagram (7.2).

Remark 7.2. Considering the number of indices to each vertex, the diagrams in the above definition can be understood as the Dynkin diagram of type C_n .

Example 7.3. For $Q = \circ_1 \xrightarrow{\quad} \circ_2 \xleftarrow{\quad} \circ_3 \xleftarrow{\quad} \circ_4$ of type A_4 , $Q^{\leftarrow 4}$ can be depicted as follows:

$$Q^{\leftarrow 4} = \circ_1 \xrightarrow{\quad} \circ_2 \xleftarrow{\quad} \circ_3 \xleftarrow{\quad} \odot_4 \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Remark 7.4. Definition 7.1 is natural in the sense that (1.2) is related to the folding operation letting n and $n+1$ be overlapped.

Definition 7.5.

- (1) We say $i \neq n, n+1$ is a *sink* (resp. *source*) of $Q^{\leftarrow n}$ or $Q^{\leftarrow n+1}$ if every arrow connecting i and another vertex j is entering into (resp. exiting out of) i .
- (2) We say $i = n$ is a *sink* (resp. *source*) of $Q^{\leftarrow n}$ if there is the arrow from $n-1$ to $\begin{pmatrix} n \\ n+1 \end{pmatrix}$ (resp. from $\begin{pmatrix} n \\ n+1 \end{pmatrix}$ to $n-1$). Similarly, $i = n+1$ is a *sink* (resp. *source*) of $Q^{\leftarrow n+1}$ if there is the arrow from $n-1$ to $\begin{pmatrix} n+1 \\ n \end{pmatrix}$ (resp. from $\begin{pmatrix} n+1 \\ n \end{pmatrix}$ to $n-1$).
- (3) A quiver $Q^{\leftarrow n}$ cannot have $n+1$ as a sink or a source. Similarly, $Q^{\leftarrow n+1}$ cannot have n as a sink or a source.

Definition 7.6. For $i \in I_{n+1}$, the *reflection function* r_i on $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$) is defined as follows.

- If $i \neq n, n+1$ and i is a sink or a source in $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$) then $i Q^{\leftarrow n}$ (resp. $i Q^{\leftarrow n+1}$) is obtained by reversing all the arrows entering into or exiting out of i .
- If $i \neq n, n+1$ and i is neither a sink nor a source then $i Q^{\leftarrow n} = Q^{\leftarrow n}$ (resp. $i Q^{\leftarrow n+1} = Q^{\leftarrow n+1}$).
- If $i = n$ (resp. $i = n+1$) then $i Q^{\leftarrow n}$ (resp. $i Q^{\leftarrow n+1}$) is obtained by (i) substituting \odot (resp. \otimes) by \otimes (resp. \odot) and (ii) reversing the arrow entering into or exiting out of i .
- If $i = n+1$ (resp. $i = n$) then $i Q^{\leftarrow n} = Q^{\leftarrow n}$ (resp. $i Q^{\leftarrow n+1} = Q^{\leftarrow n+1}$).

Now we can associate a reduced expression to a twisted Dynkin quiver.

Definition 7.7. Let $\mathbf{i} = i_1 i_2 \cdots i_{\ell(w)}$ be a reduced word of $w \in W_{n+1}^D$. We say \mathbf{i} is *adapted* to $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$) if

$$i_k \text{ is a sink of } i_{k-1} i_{k-2} \cdots i_2 i_1 Q^{\leftarrow n} \text{ (resp. } i_{k-1} i_{k-2} \cdots i_2 i_1 Q^{\leftarrow n+1})$$

for $k = 1, 2, \dots, \ell(w)$.

The following proposition shows the interesting relations between twisted Dynkin quivers and reduced expressions.

Proposition 7.8.

- (1) *There is the canonical one-to-one correspondence between the set of twisted Coxeter elements and the set of twisted Dynkin quivers. Thus we can define $\phi_{Q^{\leftarrow n}}$ and $\phi_{Q^{\leftarrow n+1}}$ in a natural way.*
- (2) *Let us denote $\phi[Q^{\leftarrow}] := \{ \phi_{Q^{\leftarrow n}}, \phi_{Q^{\leftarrow n+1}} \mid Q^{\leftarrow n} \text{ and } Q^{\leftarrow n+1} \text{ are twisted Dynkin quivers} \}$. Then*

$$\phi[Q^{\leftarrow}] = \{ \phi_{[\mathbf{i}_0]} \mid [\mathbf{i}_0] \text{ is a twisted adapted class} \}.$$

- (3) *If $[\mathbf{i}_0]$ is a twisted adapted class of reduced expressions with twisted Coxeter element $\phi_{Q^{\leftarrow n}}$ (resp. $\phi_{Q^{\leftarrow n+1}}$) then \mathbf{i}_0 is adapted to $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$).*

Proof. (1) follows by Proposition 3.5. (2) follows from the correspondence between twisted Coxeter elements and twisted Dynkin quivers proved in (1) and the correspondence between twisted Coxeter elements and classes of twisted adapted classes. Finally, (3) follows from the facts that

- (i) if $i_1 i_2$ is adapted to $\phi_{Q^{\leftarrow n}}$ (resp. $\phi_{Q^{\leftarrow n+1}}$) and $[i_1 i_2] = [i_2 i_1]$ then $i_2 i_1$ is also adapted to $\phi_{Q^{\leftarrow n}}$ (resp. $\phi_{Q^{\leftarrow n+1}}$),
- (ii) if $(s_{i_1} s_{i_2} \cdots s_{i_n})^\vee$ is the twisted Coxeter element associated to $Q^{\leftarrow n}$ (resp. $\phi_{Q^{\leftarrow n+1}}$) then $i_n \cdots i_2 i_1 Q^{\leftarrow n} = Q^{\leftarrow n+1}$ (resp. $i_n \cdots i_2 i_1 Q^{\leftarrow n+1} = Q^{\leftarrow n}$) and hence $\prod_{k=0}^n (s_{i_1} s_{i_2} \cdots s_{i_n})^{k\vee}$ is adapted to $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$).

□

Using Proposition 7.8, let us introduce simpler notations:

Definition 7.9. We denote by $\llbracket Q^{\leftarrow} \rrbracket$ the twisted adapted cluster point of D_{n+1} and denote by $[Q^{\leftarrow n}]$ (resp. $[Q^{\leftarrow n+1}]$) the class of reduced expressions adapted to $Q^{\leftarrow n}$ (resp. $Q^{\leftarrow n+1}$). Also, we denote by $\mathbf{i}_0 \in [Q^{\leftarrow}]$ when \mathbf{i}_0 is twisted adapted and denote by Q^{\leftarrow} the twisted Dynkin quiver.

With the notion of sinks and sources of Q^{\leftarrow} , we have an analogue of Lemma 4.4 for folded AR-quiver:

Lemma 7.10. *For any folded AR-quiver, positions of simple roots inside of $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ are on the boundary of $\hat{\Upsilon}_{[Q^{\leftarrow}]}$; that is, either*

- (i) α_i are a sink or a source $\hat{\Upsilon}_{[Q^{\leftarrow}]}$, or
- (ii) α_i has $\bar{1}$ or \bar{n} as its folded residue.

Furthermore,

- (a) all sinks and sources of $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ have their labels as simple roots,
- (b) i is a sink (resp. source) of Q^{\leftarrow} if and only if α_i is a sink (resp. source) of $\hat{\Upsilon}_{[Q^{\leftarrow}]}$

Proof. It is an immediate consequence of Algorithm 4.34 and Lemma 4.4 for Q of type A_n . □

Theorem 7.11. *Using Proposition 7.8 and Definition 7.9, we obtain the analogue of (2.6):*

$$(7.3) \quad \begin{array}{ccccc} & & \{\prec_{Q^{\leftarrow}}\} & & \\ & \swarrow^{1-1} & \uparrow^{1-1} & \nwarrow^{1-1} & \\ \{[Q^{\leftarrow}]\} & \xleftarrow{1-1} & \{\phi_{Q^{\leftarrow}}\} & \xrightarrow{1-1} & \{Q^{\leftarrow}\} \\ & \searrow_{1-1} & \downarrow^{1-1} & \swarrow_{1-1} & \\ & & \{\Upsilon_{[Q^{\leftarrow}]}\} & & \end{array} \text{ for } [Q^{\leftarrow}] \in \llbracket Q^{\leftarrow} \rrbracket.$$

We give a proposition about relations between Q^{\leftarrow} and $\Upsilon_{[Q^{\leftarrow}]}$ without proof.

Proposition 7.12.

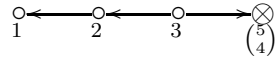
- (1) *The subquiver consisting of $\{(i, \xi(\bar{i})) \mid i = 1, \dots, n\}$ of $\Upsilon_{[Q^{\leftarrow}]}$ is isomorphic to Q^{\leftarrow} , where ξ is the height function on \bar{I} .*
- (2) *Using Q^{\leftarrow} and $\phi_{Q^{\leftarrow}}$, we can recover $\Upsilon_{[Q^{\leftarrow}]}$.*
- (3) *If $i \in I_{n+1}$ is a sink of Q^{\leftarrow} then i of Q^{\leftarrow} is associated twisted Dynkin quiver to $\Upsilon_{[Q^{\leftarrow}]} \cdot r_i$; i.e. $\Upsilon_{[iQ^{\leftarrow}]} = \Upsilon_{[Q^{\leftarrow}]} \cdot r_i$. Here $\Upsilon_{[Q^{\leftarrow}]} \cdot r_i$ is the twisted AR-quiver associated to $r_i[Q^{\leftarrow}]$.*

Now we described the reflection functor r_i on $[Q^{\leftarrow}]$ (equivalently, the map obtaining $\Upsilon_{[iQ^{\leftarrow}]}$ from $\Upsilon_{[Q^{\leftarrow}]}$) in a combinatorial way:

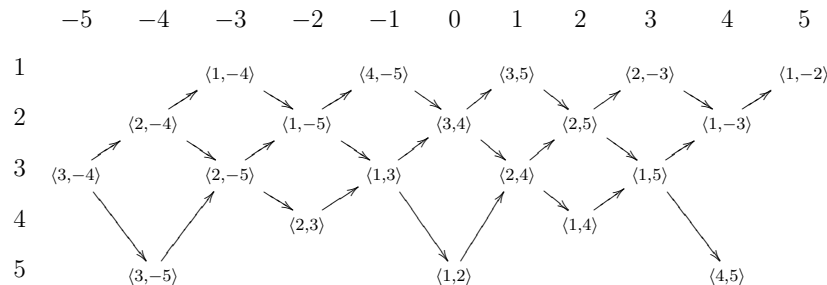
Algorithm 7.13. *Let $h^\vee = 2n + 2$ be a dual Coxeter number of D_{n+1} , an index i be a sink of $Q^{\leftarrow} \in \llbracket Q^{\leftarrow} \rrbracket$ and $*$ be the involution of D_{n+1} such that $w_0(\alpha_i) = -\alpha_{i^*}$.*

- (D1) *Remove the vertex $(i, \xi(\bar{i}))$ and the arrows entering into the vertex.*
- (D2) *Add the vertex $(i^*, \xi(\bar{i}) - h^\vee)$ and the arrows to all vertices whose coordinates are $(j, \xi(\bar{i}) - h^\vee + 1) \in \Upsilon_{[i_0]}$ where j is an index adjacent to i^* in the Dynkin diagram Δ_{n+1}^D .*
- (D3) *Label the vertex $(i^*, \xi(\bar{i}) - h^\vee)$ with α_i and change the labels β to $s_i(\beta)$ for all $\beta \in \Upsilon_{[Q^{\leftarrow}]} \setminus \{\alpha_i\}$.*

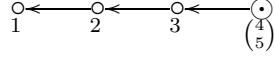
Example 7.14. Let $[i_0]$ be a twisted adapted class of D_5 with its twisted Coxeter element $(s_1 s_2 s_5 s_3)^\vee$. Then the associated twisted Dynkin quiver is $Q^{\leftarrow 5}$:



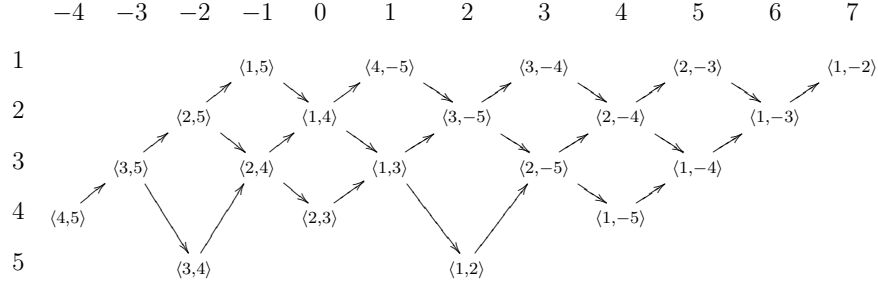
and the associated twisted AR-quiver is



Consider $Q^{\leftarrow 4} = r_5 Q^{\leftarrow 5}$. Then $Q^{\leftarrow 4}$ is



and $\Upsilon_{[Q^{\leftarrow 4}]} = \Upsilon_{[Q^{\leftarrow 5}]} \cdot r_5$ is



Let us introduce an interesting observation (see Algorithm 7.16 below).

Definition 7.15. Note that \bar{I} can be thought as an index set of C_n .

- (a) Let $\bar{D} = \text{diag}(d_{\bar{i}} \mid \bar{i} \in \bar{I}) = \text{diag}(1, 1, \dots, 1, 2)$ be the diagonal matrix which diagonalizes the Cartan matrix $A = (a_{\bar{i}\bar{j}})$ of type C_n .
- (b) We denote by $\bar{d} = \text{lcm}(d_{\bar{i}} \mid \bar{i} \in \bar{I}) = 2$.

Recall that the involution $*$ induced by the longest element w_0 of type C_n is an identity map $\bar{i} \mapsto \bar{i}$. Suppose i is a sink of Q^{\leftarrow} . We shall describe the algorithm which shows a way of obtaining $\hat{\Upsilon}_{[i Q^{\leftarrow}]}$ from $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ by using the notations on C_n . (cf. Algorithm 7.13.)

Algorithm 7.16. Let $\mathfrak{h}_C^\vee = n + 1$ be a dual Coxeter number of C_n and α_i be a sink of $[i_0] \in \llbracket Q^{\leftarrow} \rrbracket$.

- (D1) Remove the vertex $(\bar{i}, \xi(\bar{i}))$ and the arrows entering into $(\bar{i}, \xi(\bar{i}))$ in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$.
- (D2) Add the vertex $(\bar{i}, \xi(\bar{i}) - \bar{d} \times \mathfrak{h}_C^\vee)$ and the arrows to all vertices whose coordinates are $(\bar{j}, \xi(\bar{i}) - \bar{d} \times \mathfrak{h}_C^\vee + 1) \in \hat{\Upsilon}_{[Q^{\leftarrow}]}$, where \bar{j} is adjacent to \bar{i}^* in Dynkin diagram of type C_n .
- (D3) Label the vertex $(\bar{i}, \xi(\bar{i}) - \bar{d} \times \mathfrak{h}_C^\vee)$ with α_i and change the labels β to $s_i(\beta)$ for all $\beta \in \hat{\Upsilon}_{[Q^{\leftarrow}]} \setminus \{\alpha_i\}$.

8. DISTANCES AND RADIUS WITH RESPECT TO $\llbracket Q^{\leftarrow} \rrbracket$

8.1. Notions. In this subsection, we briefly review the notions on sequences of positive roots which were mainly introduced in [18, 22].

Convention 8.1. Let us choose any reduced expression $\mathbf{j}_0 = i_1 i_2 \cdots i_N$ of $w_0 \in W$ of any finite type and fix a convex total order $\leq_{\mathbf{j}_0}$ induced by \mathbf{j}_0 and a labeling of Φ^+ as follows:

$$\beta_k^{\mathbf{j}_0} := s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \in \Phi^+, \quad \beta_k^{\mathbf{j}_0} \leq_{\mathbf{j}_0} \beta_l^{\mathbf{j}_0} \text{ if and only if } k \leq l.$$

- (i) We identify a sequence $\underline{m}_{\mathbf{j}_0} = (m_1, m_2, \dots, m_N) \in \mathbb{Z}_{\geq 0}^N$ with

$$(m_1 \beta_1^{\mathbf{j}_0}, m_2 \beta_2^{\mathbf{j}_0}, \dots, m_N \beta_N^{\mathbf{j}_0}) \in (\mathbb{Q}^+)^N,$$

where \mathbb{Q}^+ is the positive root lattice.

- (ii) For a sequence \underline{m}_{j_0} and another reduced expression \mathbf{j}_0' of w_0 , the sequence $\underline{m}_{j_0'} \in \mathbb{Z}_{\geq 0}^N$ is induced from \underline{m}_{j_0} as follows : (a) Consider \underline{m}_{j_0} as a sequence of positive roots, (b) rearranging with respect to $<_{j_0'}$, (c) applying the convention (i).

For simplicity of notations, we usually drop the script \mathbf{j}_0 if there is no fear of confusion.

Definition 8.2. [22, Definition 1.10]

- (i) A sequence \underline{m} is called a *pair* if $|\underline{m}| := \sum_{i=1}^N m_i = 2$ and $m_i \leq 1$ for $1 \leq i \leq N$.
(ii) The *weight* $\text{wt}(\underline{m})$ of a sequence \underline{m} is defined by $\sum_{i=1}^N m_i \beta_i \in \mathbb{Q}^+$.

We mainly use the notation \underline{p} for a pair. We also write \underline{p} as $(\alpha, \beta) \in \Phi^+$ or $(\underline{p}_{i_1}, \underline{p}_{i_2})$ where $\beta_{i_1} = \alpha$, $\beta_{i_2} = \beta$ and $i_1 < i_2$.

Definition 8.3. [18, 22] We define partial orders $<_{j_0}^b$ and $\prec_{[j_0]}^b$ on $\mathbb{Z}_{\geq 0}^N$ as follows:

- (i) $<_{j_0}^b$ is the bi-lexicographical order induced by $<_{j_0}$.
(ii) For sequences \underline{m} and \underline{m}' , $\underline{m} \prec_{[j_0]}^b \underline{m}'$ if and only if $\text{wt}(\underline{m}) = \text{wt}(\underline{m}')$ and $\underline{m}_{j_0'} \prec_{j_0'}^b \underline{m}'_{j_0'}$ for all $\mathbf{j}_0' \in [j_0]$.

Definition 8.4. [22, Definition 1.13, Definition 1.14]

- (1) A pair \underline{p} is called $[j_0]$ -simple if there exists no sequence $\underline{m} \in \mathbb{Z}_{\geq 0}^{\ell(w)}$ such that $\underline{m} \prec_{[j_0]}^b \underline{p}$.
(2) A sequence $\underline{m} = (m_1, m_2, \dots, m_{\ell(w)}) \in \mathbb{Z}_{\geq 0}^{\ell(w)}$ is called $[j_0]$ -simple if $|\underline{m}| = 1$ or all pairs $(\underline{p}_{i_1}, \underline{p}_{i_2})$ such that $m_{i_1}, m_{i_2} > 0$ are $[j_0]$ -simple pairs.

Definition 8.5. [22, Definition 1.15] For a given $[j_0]$ -simple sequence $\underline{s} = (s_1, \dots, s_N) \in \mathbb{Z}_{\geq 0}^N$, we say that a sequence $\underline{m} \in \mathbb{Z}_{\geq 0}^N$ is called a $[j_0]$ -minimal sequence of \underline{s} if (i) $\underline{s} \prec_{j_0}^b \underline{m}$ and (ii) there exists no sequence $\underline{m}' \in \mathbb{Z}_{\geq 0}^N$ such that

$$\underline{s} \prec_{[j_0]}^b \underline{m}' \prec_{[j_0]}^b \underline{m}.$$

Definition 8.6. [22, Definition 4.8]

- (1) We say that a sequence \underline{m} has *generalized $[j_0]$ -distance k* ($k \in \mathbb{Z}_{\geq 0}$), denoted by $\text{gdist}_{[j_0]}(\underline{m}) = k$, if \underline{m} is *not* $[j_0]$ -simple and satisfies the following properties:

- (i) there exists a set of non $[j_0]$ -simple sequences $\{\underline{m}^{(s)} \mid 1 \leq s \leq k\}$ such that

$$(8.1) \quad \underline{m}^{(1)} \prec_{[j_0]}^b \dots \prec_{[j_0]}^b \underline{m}^{(k)} = \underline{m},$$

- (ii) the set of non $[j_0]$ -simple sequences $\{\underline{m}^{(s)}\}$ has the maximal cardinality among sets of sequences satisfying (8.1).

- (2) If \underline{m} is $[j_0]$ -simple, we define $\text{gdist}_{[j_0]}(\underline{m}) = 0$.

Remark 8.7. For any pair $\{\alpha, \beta\} \in (\Phi^+)^2$, we also use the notation $\text{gdist}_{[j_0]}(\alpha, \beta)$ in a natural way; that is,

$$\text{gdist}_{[j_0]}(\alpha, \beta) = \begin{cases} \text{gdist}_{[j_0]}(\alpha, \beta) & \text{if } \alpha \prec_{[j_0]}^b \beta, \\ \text{gdist}_{[j_0]}(\beta, \alpha) & \text{if } \beta \prec_{[j_0]}^b \alpha, \\ 0 & \text{if } \alpha \text{ and } \beta \text{ is incomparable with respect to } \prec_{[j_0]}^b. \end{cases}$$

Definition 8.8. [22, Definition 1.19] For a non-simple positive root $\gamma \in \Phi^+ \setminus \Pi$, the $[\mathbf{j}_0]$ -radius of γ , denoted by $\text{rds}_{[\mathbf{j}_0]}(\gamma)$, is the integer defined as follows:

$$\text{rds}_{[\mathbf{j}_0]}(\gamma) = \max(\text{gdist}_{[\mathbf{j}_0]}(\underline{p}) \mid \gamma \prec_{[\mathbf{j}_0]}^b \underline{p}).$$

Definition 8.9. [22, Definition 1.21] For a pair \underline{p} , the $[\mathbf{j}_0]$ -socle of \underline{p} , denoted by $\text{soc}_{[\mathbf{j}_0]}(\underline{p})$, is a $[\mathbf{j}_0]$ -simple sequence \underline{s} such that

$$\underline{s} \preceq_{[\mathbf{j}_0]}^b \underline{p},$$

if such \underline{s} exists uniquely.

8.2. Distances and radius for twisted adapted cluster points. Before we investigate twisted adapted cluster points, we recall the following theorem which is about adapted cluster points.

Theorem 8.10. [21, 22] Let Q be any Dynkin quiver of type AD_m .

- (1) For any pair $(\alpha, \beta) \in (\Phi^+)^2$, we have $\text{gdist}_Q(\alpha, \beta) \leq \max\{\mathbf{m}(\gamma) \mid \gamma = \alpha, \beta\}$.
- (2) For any $\gamma \in \Phi^+ \setminus \Pi$, we have

$$\text{rds}_Q(\gamma) = \mathbf{m}(\gamma).$$

- (3) For any \underline{p} , $\text{soc}_Q(\underline{p})$ is well-defined.

In this section we prove the following theorem:

Theorem 8.11. Take any $[Q^\leftarrow] \in \llbracket Q^\leftarrow \rrbracket$ of type D_{n+1} .

- (1) For any pair $(\alpha, \beta) \in (\Phi^+)^2$, we have $\text{gdist}_{[Q^\leftarrow]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\gamma) \mid \gamma = \alpha, \beta\} = 2$.
- (2) For any $\gamma \in \Phi^+ \setminus \Pi$,

$$(8.2) \quad \text{rds}_{[Q^\leftarrow]}(\gamma) = \overline{\mathbf{m}}(\gamma).$$

- (3) For any \underline{p} , $\text{soc}_{[Q^\leftarrow]}(\underline{p})$ is well-defined.

Proposition 8.12. Let $\gamma \in \Phi^+ \setminus \Pi$ and $\gamma \in \Upsilon_{[Q^\leftarrow]}^L \cup \Upsilon_{[Q^\leftarrow]}^R$. Then

$$\text{rds}_{[Q^\leftarrow]}(\gamma) = \overline{\mathbf{m}}(\gamma) = 1.$$

Proof. Note that $\gamma \in \Upsilon_{[Q^\leftarrow]}^L \cup \Upsilon_{[Q^\leftarrow]}^R$ is (folded) multiplicity free by Lemma 4.36. Thus any pairs $\underline{p} = (\alpha, \beta)$ and $\underline{p}' = (\alpha', \beta')$ with $\text{wt}(\underline{p}) = \text{wt}(\underline{p}') = \gamma$ consist of multiplicity free positive roots α, α', β and β' . By replacing α_{n+1} to α_n , we can consider $\gamma, \alpha, \alpha', \beta$ and β' as positive roots of type A_n . Assume that $\gamma \in \Upsilon_{[Q^\leftarrow]}^R$. By the convexity of $\prec_{[Q^\leftarrow]}$, α and α' are also contained in $\Upsilon_{[Q^\leftarrow]}^R$. Then our assertion follows from Algorithm 4.34 and the fact that $\text{rds}_Q(\gamma) = 1$ ([22, Theorem 4.15]) for $Q = \mathbf{p}_{A_n}^{D_{n+1}}([Q^\leftarrow])$. For the case when $\gamma \in \Upsilon_{[Q^\leftarrow]}^L$, we can apply the same argument. \square

Proposition 8.13. For $\gamma \in \Upsilon_{[Q^\leftarrow]}^C$, we have

$$\text{rds}_{[Q^\leftarrow]}(\gamma) = \overline{\mathbf{m}}(\gamma) = 2.$$

Proof. Note that $\gamma \in \Upsilon_{[Q^{\leftarrow}]}^C$ is not folded multiplicity free by Lemma 4.36. Thus

$$\gamma = \langle a, b \rangle \quad \text{for some } 1 \leq a < b \neq n+1.$$

We have $\gamma = \langle a, n+1 \rangle + \langle b, -n-1 \rangle = \langle a, -n-1 \rangle + \langle b, n+1 \rangle$. Vertices $\langle a, -n-1 \rangle$ and $\langle b, -n-1 \rangle$ are connected by a sectional path which is induced from the path between $[a, n]$ and $[b, n]$ in Γ_Q for $Q = \mathbf{p}_{A_n}^{D_{n+1}}([Q^{\leftarrow}])$. Also, we can apply the same argument for the pair consisting of $\langle a, n+1 \rangle$ and $\langle b, n+1 \rangle$. Hence $\langle a, -n-1 \rangle$ and $\langle b, -n-1 \rangle$ (resp. $\langle a, n+1 \rangle$ and $\langle b, n+1 \rangle$) are always comparable with respect to $\prec_{[Q^{\leftarrow}]}$ and

$$\langle a, -n-1 \rangle \prec_{[Q^{\leftarrow}]} \text{ (resp. } \succ_{[Q^{\leftarrow}]} \langle b, -n-1 \rangle) \iff \langle a, n+1 \rangle \prec_{[Q^{\leftarrow}]} \text{ (resp. } \succ_{[Q^{\leftarrow}]} \langle b, n+1 \rangle).$$

Thus we proved $(\langle a, n+1 \rangle, \langle b, -n-1 \rangle)$ and $(\langle a, -n-1 \rangle, \langle b, n+1 \rangle)$ are comparable with respect to $\prec_{[Q^{\leftarrow}]}^b$ and hence $\text{rds}_{[Q^{\leftarrow}]}(\gamma) \geq 2$ by the convexity of $\prec_{[Q^{\leftarrow}]}$.

Conversely, let us show that $\text{rds}_{[Q^{\leftarrow}]}(\gamma) \leq 2$. Suppose not. Then there are c and d such that

- (1) $|c| \neq |d| > b$,
- (2) $\underline{p}_1 = (\langle a, c \rangle, \langle b, -c \rangle)$ and $\underline{p}_2 = (\langle a, d \rangle, \langle b, -d \rangle)$ are comparable with respect to $\prec_{[Q^{\leftarrow}]}$.

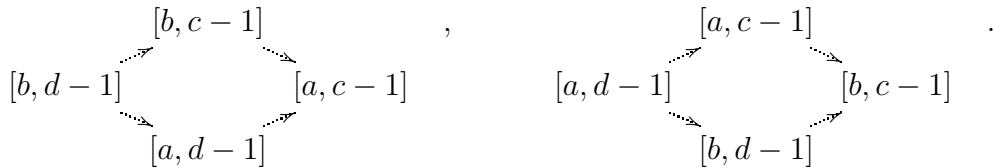
Here we assume further that

$$(i) \ c \neq d > 0, (ii) \ \underline{p}_1 \succ_{[Q^{\leftarrow}]}^b \underline{p}_2 \text{ and } (iii) \ \langle a, c \rangle \succ_{[Q^{\leftarrow}]} \langle b, -c \rangle,$$

since other cases can be proved similarly. By (ii) and (iii), we have

$$\langle a, c \rangle \succ_{[Q^{\leftarrow}]} \langle a, d \rangle, \langle b, -d \rangle \succ_{[Q^{\leftarrow}]} \langle b, -c \rangle.$$

Hence the convexity of $\prec_{[Q^{\leftarrow}]}$ tells that two roots $\langle b, -d \rangle$ and $\langle b, -c \rangle$ should be in $\Upsilon_{[Q^{\leftarrow}]}^R$. Moreover, the assumption $\langle a, c \rangle \succ_{[Q^{\leftarrow}]} \langle a, d \rangle, \langle b, -d \rangle \succ_{[Q^{\leftarrow}]} \langle b, -c \rangle$ implies that we can find one of the following rectangles (not necessarily squares) in ${}_1\Gamma_Q$ with vertices $[a, d-1]$, $[a, c-1]$, $[b, d-1]$ and $[b, c-1]$:



In both cases, Algorithm 4.34 and (ii) in Remark 4.35 tell that

- $\langle a, -c \rangle$ and $\langle a, -d \rangle$ are contained in $\Upsilon_{[Q^{\leftarrow}]}^R$,
- there exists a path $\langle a, d \rangle \rightarrow \langle a, c \rangle$ in ${}_2\Gamma_{Q^*} \subset \Upsilon_{[Q^{\leftarrow}]}$.

Hence \underline{p}_1 and \underline{p}_2 are not comparable with respect to $\prec_{[Q^{\leftarrow}]}^b$. Thus our assertion follows. \square

The first step for Theorem 8.11. From the above three lemmas, the second assertion of Theorem 8.11 follows. Furthermore, the first and the third assertions for $\alpha + \beta \in \Phi^+$ also hold. \square

Proposition 8.14. *Let both α and β be positive roots in $\Upsilon_{[Q^{\leftarrow}]}^L$ (resp. $\Upsilon_{[Q^{\leftarrow}]}^R$). Then*

$$\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\alpha), \overline{\mathbf{m}}(\beta)\} = 1.$$

Proof. It follows from [22, Theorem 4.15] and Algorithm 4.34. \square

Proposition 8.15. *If two positive roots α and β are in $\Upsilon_{[Q^{\leftarrow}]}^C$ then*

$$\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\alpha), \overline{\mathbf{m}}(\beta)\} = 2.$$

Proof. Note that

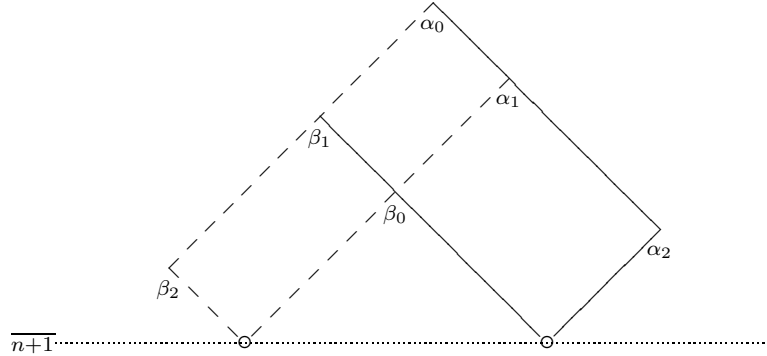
- every vertex in $\Upsilon_{[Q^{\leftarrow}]}^C$ has α_n and α_{n+1} as its support,
- if $\alpha \prec_{[Q^{\leftarrow}]} \gamma \prec_{[Q^{\leftarrow}]} \beta$ then $\gamma \in \Upsilon_{[Q^{\leftarrow}]}^C$.

Since we are only interested in sequences with the same weight as $\alpha + \beta$, any sequence whose sum is $\alpha + \beta$ should be a pair. Thus it is enough to consider increasing sequences of pairs with weight $\text{wt}(\underline{p})$

$$(8.3) \quad \underline{p}_1 \prec_{[Q^{\leftarrow}]}^b \underline{p}_2 \prec_{[Q^{\leftarrow}]}^b \cdots \prec_{[Q^{\leftarrow}]}^b \underline{p}_m = \underline{p} = (\alpha, \beta).$$

Suppose α and β are not in the same sectional. Let $\alpha = \langle a_1, a_2 \rangle$ and $\beta = \langle b_1, b_2 \rangle$ for distinct $a_1, a_2, b_1, b_2 \in \{1, 2, \dots, n\}$. A pair (α', β') with same weight as \underline{p} has a_1, a_2, b_1 and b_2 as summands of α' or β' . Hence it is enough to consider intersections of swings containing α or β .

(8.4)



Now, the pair $\underline{p} = (\alpha, \beta)$ should be one of (α_0, β_0) , (α_1, β_1) and (α_2, β_2) and we have

$$\text{gdist}_{[Q^{\leftarrow}]}(\alpha_i, \beta_i) = i$$

If α and β are in the same sectional path, there is no other pair whose weight is the same as $\alpha + \beta$ by Theorem 5.8 and the definition of $\prec_{[Q^{\leftarrow}]}^b$. \square

Lemma 8.16. *Let α be in $\Upsilon_{[Q^{\leftarrow}]}^C$ and β be in $\Upsilon_{[Q^{\leftarrow}]}^R$ or $\Upsilon_{[Q^{\leftarrow}]}^L$. If $\alpha + \beta \notin \Phi^+$ then there is no sequence \underline{m} such that*

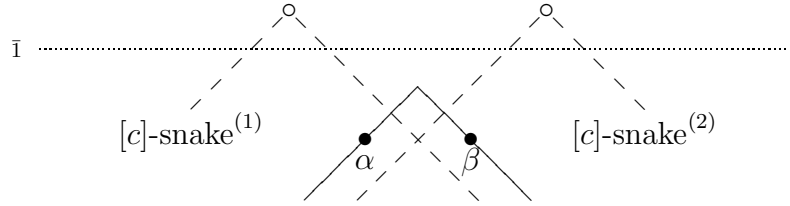
- the weight of \underline{m} is same as $\alpha + \beta$,
- $\underline{m} \prec_{[Q^{\leftarrow}]}^b (\alpha, \beta)$,
- \underline{m} is not a pair.

Proof. Consider $\beta \in \Upsilon_{[Q^{\leftarrow}]}^R$ and denote $\alpha = \langle a_1, a_2 \rangle$ and $\beta = \langle b_1, b_2 \rangle$. Note that $b_2 \in \{-2, -3, \dots, -n-1\} \cup \{n+1\}$. If $|b_2| < a_1$, the proposition directly follows from the

convexity of $\prec_{[Q \leftarrow]}$. Hence we only consider the case when $|b_2| \geq a_1$. Also, we note that it is enough to show when \underline{m} is a triple.

Let us assume that $|b_2| > a_1$. Suppose there is a triple $\underline{m} = (\gamma, \delta, \eta)$ where $(\alpha, \beta) \succ_{[Q \leftarrow]}^b \underline{m}$ and the weight is same as that of $\alpha + \beta$. The set of summands of γ, δ and η is $\{a_1, a_2, b_1, b_2, c, -c\}$ for $c \in \{2, \dots, n+1\} \setminus \{a_1, a_2, b_1, b_2\}$. Without loss of generality, let γ have c as a summand and let δ have $-c$ as a summand.

(Case I) If the N -sectional path passing α and the S -sectional path passing β have an intersection then the $[c]$ -snake in $\Upsilon_{[Q \leftarrow]}$ looks like one of $[c]$ -snake⁽¹⁾ and $[c]$ -snake⁽²⁾ in the following picture.



If the $[c]$ -snake is of the form of $[c]$ -snake⁽¹⁾ then there is no root having c as a summand between α and β . On the other hand, if the $[c]$ -snake is of the form of $[c]$ -snake⁽²⁾ then there is no root having $-c$ as a summand between α and β . Hence, in any case, we cannot find both γ and δ .

(Case II) Suppose N -sectional path passing α and S -sectional path passing β do not have an intersection. This assumption implies $|a_2| \geq |b_2|$, so that there is no root having a_2 and b_2 as summands. In this case,

- α is the intersection between $N[a_2]$ -sectional path and $S[a_1]$ -sectional path.
- β is the intersection between $N[b_1]$ -sectional path and $S[b_2]$ -sectional path.

Also, from the fact that $\beta \prec_{[Q \leftarrow]} \gamma, \delta \prec_{[Q \leftarrow]} \alpha$, we can see that

- $N[c]$ -sectional path and $S[-c]$ -sectional path should be in between $N[a_2]$ -sectional path and $S[b_2]$ -sectional path.
- $(\gamma, \delta) = (\langle a_1, c \rangle, \langle b_1, -c \rangle)$ or $(\langle a_1, -c \rangle, \langle b_1, c \rangle)$.

However, since $|a_2| \geq |b_2|$, there is no root of the form $\langle a_2, b_2 \rangle$. Hence $\alpha + \beta - \gamma - \delta = \eta \notin \Phi^+$, which is a contradiction.

For the case $|b_2| = a_1$, we omit the proof since it can be proved similarly. Moreover, the same argument works when $\beta \in \Upsilon_{[Q \leftarrow]}^L$. \square

Proposition 8.17. *Let α be in $\Upsilon_{[Q \leftarrow]}^C$ and β be in $\Upsilon_{[Q \leftarrow]}^R$ or $\Upsilon_{[Q \leftarrow]}^L$. If $\alpha + \beta \notin \Phi^+$ then*

$$\text{gdist}_{[Q \leftarrow]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\alpha), \overline{\mathbf{m}}(\beta)\} = \overline{\mathbf{m}}(\alpha) = 2.$$

Proof. By Lemma 8.16, it is enough to consider pairs \underline{p} whose weight are same as that of $\alpha + \beta$. Now, we can apply the same argument as in the proof of Proposition 8.15 which was described in (8.4). \square

Proposition 8.18. *Let α be in one of $\Upsilon_{[Q^{\leftarrow}]}^L$ and $\Upsilon_{[Q^{\leftarrow}]}^R$, and β be in the other. If $\alpha + \beta \notin \Phi^+$ then*

$$\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) \leq \max\{\overline{\mathbf{m}}(\alpha), \overline{\mathbf{m}}(\beta)\} = 1.$$

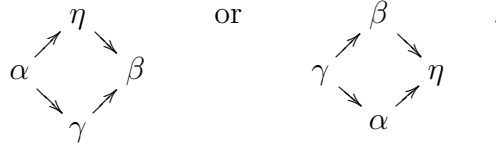
Proof. Note that α, β are (folded) multiplicity free and hence can be identified with corresponding positive roots in $\Phi_{A_n}^+$ in a natural way. Without loss of generality, assume that $\alpha \in \Upsilon_{[Q^{\leftarrow}]}^L$ and $\beta \in \Upsilon_{[Q^{\leftarrow}]}^R$. Let us denote $\alpha = \langle a_1, -a_2 \rangle$ and $\beta = \langle b_1, -b_2 \rangle$. Then the followings are true.

- If $b_1 - 1 > a_2$ or $a_1 - 1 > b_2$ then there is no other pair with the same weight and hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 0$ ([22, (4.2)]).
- If $b_1 = -a_2$ or $a_1 = -b_2$ then $\alpha + \beta \in \Phi^+$.
- If $|a_2| = |b_2| = n + 1$ then $\alpha + \beta \notin \Phi^+$ implies $a_1 = b_1$. In this case, one can check that $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 0$.

For the other cases, there exists another pair $\{\gamma, \eta\}$ such that

$$\alpha + \beta = \gamma + \eta \notin \Phi^+.$$

Note that such a pair $\{\gamma, \eta\}$ is unique. There is a square in the AR-quiver Γ_Q for $Q = \mathbf{p}_{A_n}^{D_{n+1}}([Q^{\leftarrow}])$ which is one of the followings (Theorem 4.3 and [20, Proposition 4.12]):



Here $\eta = \langle a_1, -b_2 \rangle$ and $\gamma = \langle b_1, -a_2 \rangle$ by Theorem 4.3 and they are also (folded) multiplicity free. Recall Remark 4.35 that (i) there is a path from any vertex in $\Upsilon_{[Q^{\leftarrow}]}^L$ to any vertex in $\Upsilon_{[Q^{\leftarrow}]}^R$ and (ii) η (resp. γ) is in $\Upsilon_{[Q^{\leftarrow}]}^R$ (resp. $\Upsilon_{[Q^{\leftarrow}]}^L$). For the first case, we get the path $\alpha \rightarrow \gamma \rightarrow \eta \rightarrow \beta$ by Algorithm 4.34 and the assumption of our assertion. For the second case, we get the path $\gamma \rightarrow \alpha \rightarrow \beta \rightarrow \eta$ by the same reason. Hence we get $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ and $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 0$, respectively. For a non-pair sequence \underline{m} such that $\text{wt}(\underline{m}) = \alpha + \beta$, it cannot be $\underline{m} \prec_{[Q^{\leftarrow}]} (\alpha, \beta)$ by the convexity of $\prec_{[Q^{\leftarrow}]}$. The remained cases can be proved by applying the similar argument. \square

The second step for Theorem 8.11. By Proposition 8.14, 8.15, 8.17 and 8.18, we complete Theorem 8.11. \square

Remark 8.19. Note that, for a pair (α, β) with $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 2$, there exists a unique sequence \underline{p}' , which is a pair, with $\text{wt}(\underline{p}') = \alpha + \beta$ such that

$$\text{soc}_{[Q^{\leftarrow}]}(\alpha, \beta) \prec_{[Q^{\leftarrow}]}^b \underline{p}' \prec_{[Q^{\leftarrow}]}^b (\alpha, \beta).$$

However, in an adapted class $[Q]$ of type D , every pair (α, β) with $\text{gdist}_{[Q]}(\alpha, \beta) = 2$, there exist sequences $\underline{m}^{(1)}$ and $\underline{m}^{(2)}$ such that

- $\text{wt}(\underline{m}^{(1)}) = \text{wt}(\underline{m}^{(2)}) = \alpha + \beta$,
- they are incomparable with respect to $\prec_{[Q]}^b$,

- $\text{soc}_{[Q]}(\alpha, \beta) \prec_{[Q \leftarrow]}^b \frac{\underline{m}^{(1)}}{\underline{m}^{(2)}} \prec_{[Q]}^b (\alpha, \beta)$.

Theorem 8.20. *Let α and β have coordinates (\bar{i}, p) and (\bar{j}, q) in the folded AR-quiver $\hat{\Upsilon}_{[Q \leftarrow]}$. Then the $\text{gdist}_{[Q \leftarrow]}(\alpha, \beta)$ depends only on \bar{i}, \bar{j} and $|p - q|$.*

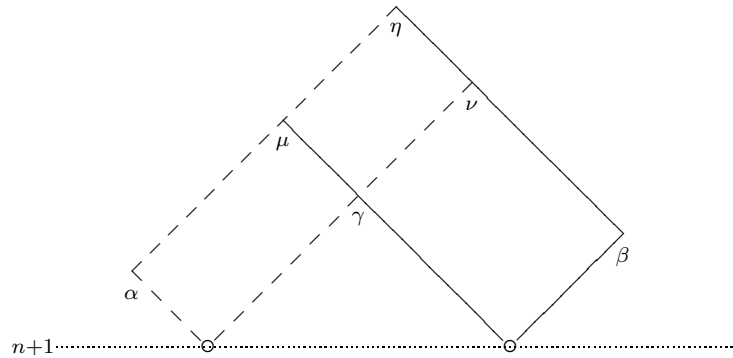
Proof. Let us assume that $\alpha \succ_{[Q \leftarrow]} \beta$. Then $q > p$. Depending on whether there exists an intersection of the S -sectional path passing α and the N -sectional path passing β in $\hat{\Upsilon}_{[Q \leftarrow]}$, we classify the type of the pair (α, β) :

- (α, β) is *Type I* if there exists no intersection between the S -sectional path passing α and the N -sectional path passing β .
- (α, β) is *Type II* if there exists an intersection between the two sectional paths.

(Type I) the picture in (8.5) is induced from four swings containing α and β . Note that μ, ν, γ , and η are not necessarily in $\hat{\Upsilon}_{[Q \leftarrow]}$ but they are on the rays containing corresponding sectional paths. In order to give coordinates to μ, ν, γ, η which are not in $\hat{\Upsilon}_{[Q \leftarrow]}$, we extend the coordinate in a “canonical” way. The new coordinate in $\{(k, r) \in \mathbb{Z} \times \mathbb{Z} \mid k \leq n+1\}$ has the following properties:

- For $k' \in \{1, 2, \dots, n\}$, if a vertex has the coordinate (\bar{k}', r') in $\hat{\Upsilon}_{[Q \leftarrow]}$ then, in the new coordinate, it has the new coordinate (k', r') .
- For $k' \in \{n+1\} \cup \{0, -1, -2, \dots\}$, the coordinate (k', r') is the intersection of two rays containing the following sectional paths :
 - N -sectional path consisting of $(\overline{k' + l}, r' + l) \cap \hat{\Upsilon}_{[Q \leftarrow]}$ for $l \in \mathbb{Z}$,
 - S -sectional path consisting of $(\overline{k' - l}, r' + l) \cap \hat{\Upsilon}_{[Q \leftarrow]}$ for $l \in \mathbb{Z}$.

(8.5)



More precisely, we can compute the coordinates as follows.

$$\begin{aligned}
 \eta &= (\eta_0, \eta_1) = ((i + j + p - q)/2, i - j + p + q), \\
 \mu &= (\mu_0, \mu_1) = (n + 1 + (i - j + p - q)/2, \eta_1 + j - n - 1), \\
 \nu &= (\nu_0, \nu_1) = (n + 1 + (j - i + p - q)/2, \eta_1 + i - n - 1), \\
 \gamma &= (\gamma_0, \gamma_1) = (2n + 2 - (i + j - p + q)/2, \mu_1 + \nu_1 - \eta_1).
 \end{aligned}
 \tag{8.6}$$

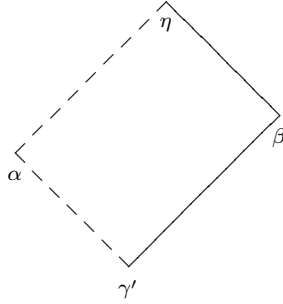
Now we note that $\eta_0, \mu_0, \nu_0, \gamma_0$ depend only on i, j and $|p - q|$. Moreover, by Proposition 8.15, Proposition 8.17 and Proposition 8.18, we have

(8.7)

$$\begin{aligned} \text{gdist}_{[Q \leftarrow]}(\alpha, \beta) &= 2 \quad \text{if } \eta_0 \geq 0 \text{ and } \gamma_0 < n + 1; \\ \text{gdist}_{[Q \leftarrow]}(\alpha, \beta) &= 1 \quad \text{if (i) } (\eta_0, \gamma_0) \notin (\mathbb{Z}_{\geq 0}, \mathbb{Z}_{< n+1}), \text{ (ii) } (\mu_0, \nu_0) \in \mathbb{Z}_{> 0} \times \mathbb{Z}_{\geq 0} \cup \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{> 0}; \\ \text{gdist}_{[Q \leftarrow]}(\alpha, \beta) &= 0 \quad \text{otherwise.} \end{aligned}$$

(Type II) In this case, we have the following picture. As in (Type I), η is not necessarily in $\hat{\Upsilon}_{[Q \leftarrow]}$. However, γ' is in $\hat{\Upsilon}_{[Q \leftarrow]}$.

(8.8)



Here, η has the same coordinate as in (8.6) and

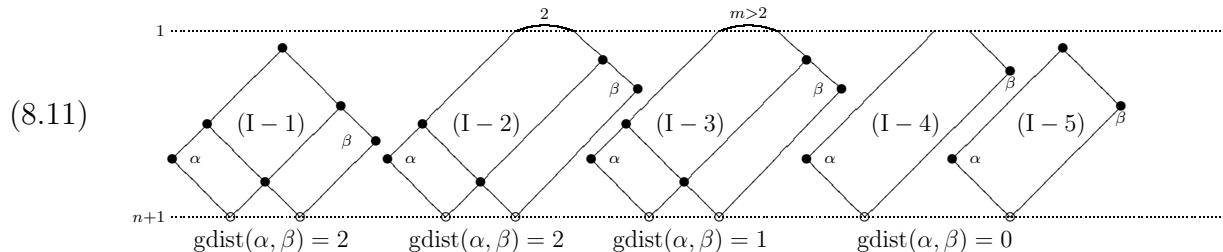
$$(8.9) \quad \gamma' = (\gamma'_0, \gamma'_1) = ((i + j - p + q)/2, p + q - \eta_1)$$

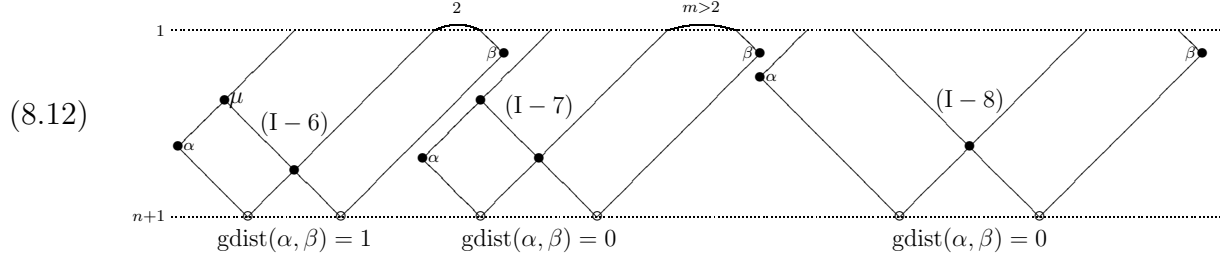
$$(8.10) \quad \begin{aligned} \text{gdist}_{[Q \leftarrow]}(\alpha, \beta) &= 1 \quad \text{if } \eta_0 \geq 0; \\ \text{gdist}_{[Q \leftarrow]}(\alpha, \beta) &= 0 \quad \text{otherwise.} \end{aligned}$$

Since $\gamma_0, \eta_0, \mu_0, \nu_0$ and γ'_0 depend only on i, j and $|p - q|$ and by (8.7) and (8.10). Moreover, the assumption $1 \leq i, j \leq n$ implies $i = \bar{i}$ and $j = \bar{j}$. Hence we conclude that $\text{gdist}_{[Q \leftarrow]}(\alpha, \beta)$ only depends on \bar{i}, \bar{j} and $|p - q|$. \square

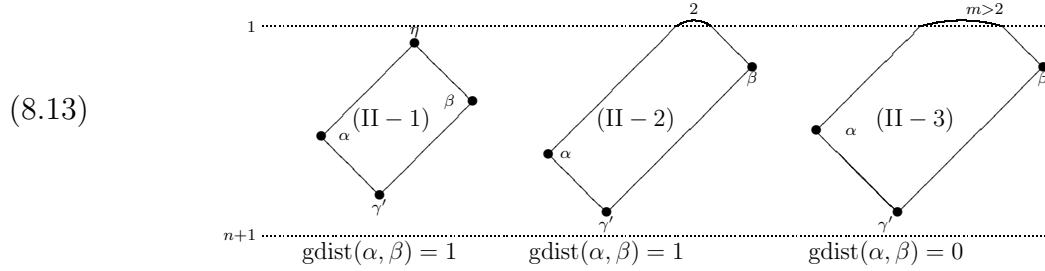
Remark 8.21. Let α, β be positive roots of D_{n+1} and the coordinates (\bar{i}, p) and (\bar{j}, q) of α and β in $\hat{\Upsilon}_{[Q \leftarrow]}$ satisfy $i \geq j$ and $q > p$.

If (α, β) is of (Type I) in the proof of Theorem 8.20 then the pair (α, β) has the form of (I-1) \sim (I-8) in (8.11) or (8.12). Note that the layer $n + 1$ in (8.11) and (8.12) is induced from the new coordinate introduced in the proof of Theorem 8.20.





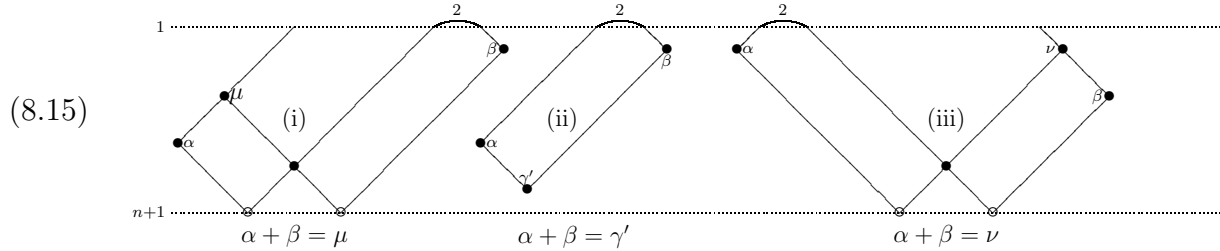
Now, let (α, β) be of (Type II) in the proof of Theorem 8.20. Then we have one of the following pictures. If we have (II-1) or (II-2) then $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$. If we have (II-3) then $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 0$



Hence Theorem 8.20 shows

$$(8.14) \quad \text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = \begin{cases} 2 & \text{if } (\alpha, \beta) \text{ is one of (I-1) or (I-2),} \\ 1 & \text{if } (\alpha, \beta) \text{ is one of (I-3), (I-6), (II-1) or (II-2),} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 8.22. Observe (I-6) and (II-2) in Remark 8.21. In these cases, we have $\alpha + \beta = \mu \in \Phi^+$ and $\alpha + \beta = \gamma' \in \Phi^+$. More precisely, Theorem 8.20 shows if $\alpha + \beta \in \Phi^+$ and $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ for $\alpha \succ_{[Q^{\leftarrow}]} \beta$ then (α, β) has one of the following pictures.



The picture (i) and (iii) are of (Type I) in Theorem 8.20. Using the notations in the proof of Theorem 8.20, the picture (i) (resp. (iii)) implies $\eta_0 < 0$, $\mu_0 > 0$ (resp. $\nu_0 > 0$) and $\nu_0 = 0$ (resp. $\mu_0 = 0$). The picture (ii) is of (Type II) in Theorem 8.20 and it implies $\eta_0 = 0$.

Remark 8.23. Let $\alpha + \beta \in \Phi^+ \setminus \Pi$. By Remark 8.21 and Remark 8.22 we can find $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta)$ using (8.5) and (8.8) by the following facts.

- (1) $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 2$ if and only if γ, η, μ, ν in (8.5) are all in $\widehat{\Upsilon}_{[Q^{\leftarrow}]}$.
- (2) $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ if and only if one of the followings hold:

- (a) not both of γ and η in (8.5) are in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ but both of μ and ν are in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$,
- (b) both γ' and η in (8.8) are in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$.

Corollary 8.24. *Suppose $[Q_1^{\leftarrow}]$ and $[Q_2^{\leftarrow}]$ are both twisted adapted classes. Let α and $\beta \in \Phi^+$ have folded coordinates (\bar{i}, p) and (\bar{j}, q) , respectively, in $\hat{\Upsilon}_{[Q_1^{\leftarrow}]}$ and let α' and $\beta' \in \Phi^+$ have folded coordinates (\bar{i}, p') and (\bar{j}, q') , respectively, in $\hat{\Upsilon}_{[Q_2^{\leftarrow}]}$. If $p - q = p' - q'$ then $\text{gdist}_{[Q_1^{\leftarrow}]}(\alpha, \beta) = \text{gdist}_{[Q_2^{\leftarrow}]}(\alpha', \beta')$.*

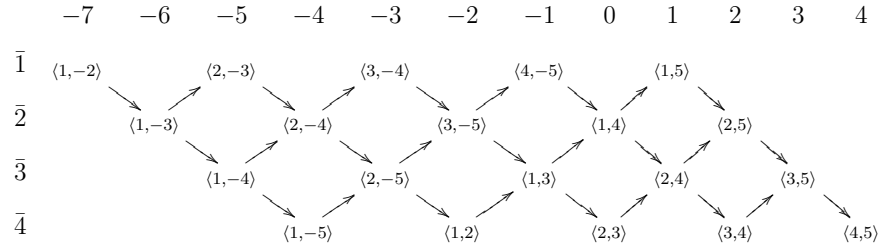
Proof. Since (8.7) and (8.10) do not depend on the classes of reduced expressions, our assertion follows. \square

Corollary 8.25. *For $\alpha, \beta, \gamma \in \Phi^+$ with $\hat{\phi}_{[Q^{\leftarrow}]}(\alpha) = (\bar{i}, p)$, $\hat{\phi}_{[Q^{\leftarrow}]}(\beta) = (\bar{j}, q)$, $\hat{\phi}_{[Q^{\leftarrow}]}(\gamma) = (\bar{k}, r)$ such that $\alpha + \beta = \gamma$, the pair (α, β) is a $[Q^{\leftarrow}]$ -minimal pair of γ if and only if one of the following conditions holds:*

$$(8.16) \quad \left\{ \begin{array}{l} \ell := \max(\bar{i}, \bar{j}, \bar{k}) \leq n, \ s + m = \ell \text{ for } \{s, m\} := \{\bar{i}, \bar{j}, \bar{k}\} \setminus \{\ell\} \text{ and} \\ (q - r, p - r) = \begin{cases} (-\bar{i}, \bar{j}), & \text{if } \ell = \bar{k}, \\ (\bar{i} - (2n + 2), \bar{j}), & \text{if } \ell = \bar{i}, \\ (-\bar{i}, 2n + 2 - \bar{j}), & \text{if } \ell = \bar{j}. \end{cases} \end{array} \right.$$

Proof. Our assertion follows from the consideration on folded coordinates of (8.15) in Remark 8.22. \square

Example 8.26. Let us consider the reduced expression $\mathbf{i}_0 \in [Q^{\leftarrow}]$ of D_5 with the twisted Coxeter element $(s_5 s_3 s_2 s_1) \vee$. Then $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ is as follows.



Let us use notations in the proof of Theorem 8.20.

- (1) Let $\alpha = \langle 1, -4 \rangle$ and $\beta = \langle 2, 3 \rangle$. Then (α, β) is Type I and we have $\mu = \langle 2, -4 \rangle$, $\nu = \langle 1, 3 \rangle$, $\delta = \langle 3, -4 \rangle$, $\gamma = \langle 1, 2 \rangle$. Hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 2$ since $(\alpha, \beta) \succ_{[Q^{\leftarrow}]}^b (\mu, \nu) \succ_{[Q^{\leftarrow}]}^b (\delta, \gamma)$.
- (2) Let $\alpha = \langle 2, -5 \rangle$ and $\beta = \langle 3, 5 \rangle$. Then (α, β) is Type I and we have $\mu = \langle 3, -5 \rangle$, $\nu = \langle 2, 5 \rangle$, and $\gamma = \langle 2, 3 \rangle$. In addition, δ is not in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ but $\delta_0 = 0$. Hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 2$ since $(\alpha, \beta) \succ_{[Q^{\leftarrow}]}^b (\mu, \nu) \succ_{[Q^{\leftarrow}]}^b \gamma$.
- (3) Let $\alpha = \langle 2, -4 \rangle$ and $\beta = \langle 3, 5 \rangle$. Then (α, β) is Type I and we have $\mu = \langle 3, -4 \rangle$, $\nu = \langle 2, 5 \rangle$, $\gamma = \langle 2, 3 \rangle$ and δ is not in $\hat{\Upsilon}_{[Q^{\leftarrow}]}$. Hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ since $(\alpha, \beta) \succ_{[Q^{\leftarrow}]}^b (\mu, \nu)$.

- (4) Let $\alpha = \langle 2, -4 \rangle$ and $\beta = \langle 1, 3 \rangle$. Then (α, β) is Type *II* and we have $\delta = \langle 3, -4 \rangle$, $\gamma' = \langle 1, 2 \rangle$. Hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ since $(\alpha, \beta) \succ_{[Q^{\leftarrow}]}^b (\delta, \gamma')$.
- (5) Let $\alpha = \langle 2, -4 \rangle$ and $\beta = \langle 1, 4 \rangle$. Then (α, β) is Type *II* and we have $\gamma' = \langle 1, 2 \rangle$ and $\delta_0 = 0$. Hence $\text{gdist}_{[Q^{\leftarrow}]}(\alpha, \beta) = 1$ since $(\alpha, \beta) \succ_{[Q^{\leftarrow}]}^b \gamma'$.

9. DISTANCE POLYNOMIAL AND FOLDED DISTANCE POLYNOMIAL

In this section, we briefly review the distance polynomials defined on the adapted cluster point $\llbracket Q \rrbracket$ of type ADE_m , which was studied in [22]. Then we introduce and study the folded distance polynomials as in [24], which are well-defined on the twisted adapted cluster point $\llbracket Q^{\leftarrow} \rrbracket$ of type D_{n+1} . The folded distance polynomials have an interesting relation with the quantum affine algebra of type $C_n^{(1)}$. This section can be understood as a twisted analogue of [22, Section 6]. In this section, we refer to and follow [22, Section 6] and [15] instead of introducing the notions for quantum affine algebras, their integrable representations and denominator formulas.

Let us take a base field \mathbf{k} the algebraic closure of $\mathbb{C}(q)$ in $\cup_{m>0} \mathbb{C}((q^{1/m}))$.

9.1. Distance polynomial.

Definition 9.1. [22, Definition 6.11] For a Dynkin quiver Q , indices $k, l \in I$ and an integer $t \in \mathbb{N}$, we define the subset $\Phi_Q(k, l)[t]$ of $\Phi^+ \times \Phi^+$ as follows:

A pair (α, β) is contained in $\Phi_Q(k, l)[t]$ if $\alpha \prec_Q \beta$ or $\beta \prec_Q \alpha$ and

$$\{\tilde{\phi}_Q(\alpha), \tilde{\phi}_Q(\beta)\} = \{(k, a), (l, b)\} \quad \text{such that} \quad |a - b| = t.$$

Lemma 9.2. [22, Lemma 6.12] For any $(\alpha^{(1)}, \beta^{(1)})$ and $(\alpha^{(2)}, \beta^{(2)})$ in $\Phi_Q(k, l)[t]$, we have

$$o_t^Q(k, l) := \text{gdist}_Q(\alpha^{(1)}, \beta^{(1)}) = \text{gdist}_Q(\alpha^{(2)}, \beta^{(2)}).$$

We denoted by Q^{rev} the quiver obtained by reversing all arrows of Q and Q^* the quiver obtained from Q by replacing vertices i of Q with i^* (see (4.5) for $*$ of type D_{n+1}).

Definition 9.3. [22, Definition 6.15] For $k, l \in I$ and a Dynkin quiver Q , we define a polynomial $D_{k,l}^Q(z) \in \mathbf{k}[z]$ as follows:

$$D_{k,l}^Q(z) := \prod_{t \in \mathbb{Z}_{\geq 0}} (z - (-1)^t q^t)^{\bar{o}_t^Q(k,l)},$$

where

$$(9.1) \quad \bar{o}_t^Q(k, l) := \max(o_t^Q(k, l), o_t^{Q^{\text{rev}}}(k, l)).$$

Proposition 9.4. [22, Proposition 6.16] For $k, l \in I$ and any Dynkin quivers Q and Q' , we have

- (a) $D_{k,l}^Q(z) = D_{l,k}^Q(z) = D_{k^*,l^*}^Q(z) = D_{l^*,k^*}^Q(z)$.
 (b) $D_{k,l}^Q(z) = D_{k,l}^{Q'}(z)$.

Hence $D_{k,l}(z)$ is well-defined for $\llbracket Q \rrbracket$.

The denominator formulas $d_{k,l}^{\mathfrak{g}}(z) = d_{l,k}^{\mathfrak{g}}(z)$ ($1 \leq k, l \leq n+1$) for $U'_q(\mathfrak{g})$ ($\mathfrak{g} = A_n^{(1)}$ or $D_{n+1}^{(1)}$) were computed in [1, 14]:

Theorem 9.5.

(a) For $\mathfrak{g} = A_n^{(1)}$ ($n \geq 2$) and $1 \leq k, l \leq n$, we have

$$(9.2) \quad d_{k,l}^{A_n^{(1)}}(z) = \prod_{s=1}^{\min(k, l, n+1-k, n+1-l)} (z - (-q)^{|k-l|+2s}).$$

(b) For $\mathfrak{g} = D_{n+1}^{(1)}$ ($n \geq 3$) and $1 \leq k, l \leq n+1$, we have

$$(9.3) \quad d_{k,l}^{D_{n+1}^{(1)}}(z) = \begin{cases} \prod_{s=1}^{\min(k,l)} (z - (-q)^{|k-l|+2s}) \prod_{s=1}^{\min(k,l)} (z - (-q)^{2n-k-l+2s}) & \text{if } 1 \leq k, l \leq n-1, \\ \prod_{s=1}^k (z - (-q)^{n-k+2s}) & \text{if } 1 \leq k \leq n-1 \text{ and } l \in \{n, n+1\}, \\ \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} (z - (-q)^{4s}) & \text{if } k \neq l \in \{n, n+1\}, \\ \prod_{s=1}^{\lfloor \frac{n+1}{2} \rfloor} (z - (-q)^{4s-2}) & \text{if } k = l \in \{n, n+1\}. \end{cases}$$

Theorem 9.6. [22, Theorem 6.18] For any adapted class $[Q]$ of type AD_m , the denominator formulas for the quantum affine algebra $U'_q(\mathfrak{g})$ ($\mathfrak{g} = A_m^{(1)}$ or $D_m^{(1)}$) can be read from Γ_Q and $\Gamma_{Q^{\text{rev}}}$ as follows:

$$d_{k,l}^{\mathfrak{g}}(z) = D_{k,l}(z) \times (z - (-q)^{\mathfrak{h}^\vee})^{\delta_{l,k^*}}$$

where \mathfrak{h}^\vee is the dual Coxeter number corresponding to Q .

9.2. Folded distance polynomial. Now we fix the folded index set

$$\bar{I} = \{1, 2, \dots, n\}$$

which is induced from \vee in (1.2).

Definition 9.7. For a folded AR-quiver $\widehat{\Upsilon}_{[Q \leftarrow]}$, indices $\bar{k}, \bar{l} \in \bar{I}$ and an integer $t \in \mathbb{N}$, we define the subset $\Phi_{[Q \leftarrow]}(\bar{k}, \bar{l})[t]$ of $\Phi^+ \times \Phi^+$ as follows:

A pair (α, β) is contained in $\Phi_{[Q \leftarrow]}(\bar{k}, \bar{l})[t]$ if $\alpha \prec_{[Q \leftarrow]} \beta$ or $\beta \prec_{[Q \leftarrow]} \alpha$ and

$$\{\widehat{\phi}_{[Q \leftarrow]}(\alpha), \widehat{\phi}_{[Q \leftarrow]}(\beta)\} = \{(\bar{k}, a), (\bar{l}, b)\} \quad \text{such that} \quad |a - b| = t.$$

By Corollary 8.24, the following notion is well-defined:

Definition 9.8. For any $(\alpha^{(1)}, \beta^{(1)}) \in \Phi_{[Q \leftarrow]}(\bar{k}, \bar{l})[t]$, we define

$$o_t^{[Q \leftarrow]}(\bar{k}, \bar{l}) := \text{gdist}_{[Q \leftarrow]}(\alpha^{(1)}, \beta^{(1)}).$$

Recall the notations on C_n in Definition 7.15.

Definition 9.9. For $\bar{k}, \bar{l} \in \bar{I}$ and a folded AR-quiver $\hat{\Upsilon}_{[Q^{\leftarrow}]}$, we define a polynomial $\hat{D}_{\bar{k}, \bar{l}}^{[Q^{\leftarrow}]}(z) \in \mathbf{k}[z]$ as follows:

$$\hat{D}_{\bar{k}, \bar{l}}^{[Q^{\leftarrow}]}(z) := \prod_{t \in \mathbb{Z}_{\geq 0}} (z - (-q_s)^t)^{\circ_t^{[Q^{\leftarrow}]}(\bar{k}, \bar{l})},$$

where

$$q_s^{\bar{d}} = q_s^2 = q \quad \text{and} \quad \circ_t^{[Q^{\leftarrow}]}(\bar{k}, \bar{l}) := \left\lfloor \frac{o_t^{[Q^{\leftarrow}]}(\bar{k}, \bar{l})}{\bar{d}} \right\rfloor.$$

Proposition 9.10. For $\bar{k}, \bar{l} \in \bar{I}$ and any twisted adapted classes $[Q_1^{\leftarrow}]$ and $[Q_2^{\leftarrow}]$ in $\llbracket Q^{\leftarrow} \rrbracket$, we have

$$\hat{D}_{\bar{k}, \bar{l}}^{[Q_1^{\leftarrow}]}(z) = \hat{D}_{\bar{k}, \bar{l}}^{[Q_2^{\leftarrow}]}(z).$$

Proof. It is an easy consequence which can be obtained from Corollary 8.24 and the fact that $\hat{\Upsilon}_{[i_0]}$ has $n+1$ -vertices in each folded residue. \square

From the above proposition, we can define the folded distance polynomial $\hat{D}_{\bar{k}, \bar{l}}(z)$ for $\llbracket Q^{\leftarrow} \rrbracket$ canonically.

Theorem 9.11. [1]

$$d_{k,l}^{C_n^{(1)}}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} (z - (-q_s)^{|k-l|+2s}) \prod_{i=1}^{\min(k,l)} (z - (-q_s)^{2n+2-k-l+2s}) \quad 1 \leq k, l \leq n$$

Theorem 9.12. For any twisted adapted class $[i_0]$, the denominator formulas for $U_q'(C_n^{(1)})$ can be read from $\hat{\Upsilon}_{[i_0]}$ as follows:

$$d_{\bar{k}, \bar{l}}^{C_n^{(1)}}(z) = \hat{D}_{\bar{k}, \bar{l}}(z) \times (z - q^{\mathbf{h}^\vee})^{\delta_{\bar{l}, \bar{k}}}$$

where $\mathbf{h}^\vee = n+1$ is the dual Coxeter number of C_n .

Proof. Note that, for $1 \leq k, l \leq n$, one can observe that

- (9.4) (i) the first factor of $d_{k,l}^{C_n^{(1)}}(z)$ is the same as $d_{k,l}^{A_{n-1}^{(1)}}(z)$,
 (ii) the second factor of $d_{k,l}^{C_n^{(1)}}(z)$ is the same as the second factor of $d_{k,l}^{D_{n+2}^{(1)}}(z)$.

Thus we can apply the same argument of [22, Theorem 6.18]. More precisely, (i) is induced from (II-1) and (II-2) in (8.13) and (ii) is induced from (I-1), (I-2), (I-3) and (I-6) in (8.11).

Alternatively, it suffices to consider a particular folded AR-quiver $\hat{\Upsilon}_{[Q^{\leftarrow}]}$ by Proposition 9.10. For the following $Q^{\leftarrow n}$, using Algorithm 4.34 and [20, Remark 1.14], one can label $\hat{\Upsilon}_{[Q^{\leftarrow n}]}$ and check our assertion (see Example 4.9):

$$Q^{\leftarrow n} = \begin{array}{c} \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \cdots \circ \longleftarrow \circ \\ 1 \qquad 2 \qquad 3 \qquad 4 \qquad \qquad n-1 \qquad \left(\begin{smallmatrix} n \\ n+1 \end{smallmatrix} \right) \end{array}$$

\square

10. DOREY'S RULE FOR $U'_q(C_n^{(1)})$

In [24, Section 9], the authors proved that Dorey's rule for $U'_q(B_{n+1}^{(1)})$ ([7]) can be interpreted the $[\mathbf{i}_0]$ -minimal pairs (α, β) of $\gamma \in \Phi_{A_{2n+1}}^+$ when $[\mathbf{i}_0]$ is a twisted adapted class of A_{2n+1} . In this section, we shall do such an analogue for $U'_q(C_n^{(1)})$ by using a twisted adapted class $[Q^\leftarrow]$ of type D_{n+1} and its folded AR-quiver $\widehat{\mathcal{T}}_{[Q^\leftarrow]}$.

Recall the quantum affine algebra $U'_q(C_n^{(1)})$ is the associative algebra with 1 generated by e_i, f_i (Chevalley generators), $K_i = q^{h_i}$ ($i \in \bar{I} \sqcup \{0\}$) subject to certain relations (see [22] for more detail).

Proposition 10.1. [7, Theorem 8.2] *Let $(\bar{i}, x), (\bar{j}, y), (\bar{k}, z) \in \bar{I} \times \mathbf{k}^\times$. Then*

$$\mathrm{Hom}_{U'_q(C_n^{(1)})}(V(\varpi_{\bar{j}})_y \otimes V(\varpi_{\bar{i}})_x, V(\varpi_{\bar{k}})_z) \neq 0$$

if and only if one of the following conditions holds:

$$(10.1) \quad \left\{ \begin{array}{l} \ell := \max(\bar{i}, \bar{j}, \bar{k}) \leq n, \ s + m = \ell \text{ for } \{s, m\} := \{\bar{i}, \bar{j}, \bar{k}\} \setminus \{\ell\} \text{ and} \\ (y/z, x/z) = \begin{cases} ((-q_s)^{-\bar{i}}, (-q_s)^{\bar{j}}), & \text{if } \ell = \bar{k}, \\ ((-q_s)^{\bar{i}-(2n+2)}, (-q_s)^{\bar{j}}), & \text{if } \ell = \bar{i}, \\ ((-q_s)^{-\bar{i}}, (-q_s)^{2n+2-\bar{j}}), & \text{if } \ell = \bar{j}. \end{cases} \end{array} \right.$$

Here,

- $V(\varpi_i)$ is the unique simple $U'_q(C_n^{(1)})$ -module which is finite dimensional integrable with its dominant weight ϖ_i ($i \in \bar{I}$) (see [15] for more detail),
- $V(\varpi_i)_x$ is $V(\varpi_i)$ as a vector space with the actions of e_i, f_i, K_i ($i \in \bar{I} \sqcup \{0\}$) replaced with $x^{\delta_{i0}}e_i, x^{-\delta_{i0}}f_i, K_i$, respectively.

Definition 10.2. For any positive root β contained in $\Phi_{D_{n+1}}^+$, we set the $U'_q(C_n^{(1)})$ -module $V_{[Q^\leftarrow]}(\beta)$ defined as follows:

$$(10.2) \quad V_{[Q^\leftarrow]}(\beta) := V(\varpi_{\bar{i}})_{(-q_s)^p} \quad \text{where} \quad \widehat{\phi}(\beta) = (\bar{i}, p).$$

We define the smallest abelian full subcategory $\mathcal{C}_{[Q^\leftarrow]}$ consisting of finite dimensional integrable $U'_q(C_n^{(1)})$ -modules such that

- it is stable by taking subquotient, tensor product and extension,
- it contains $V_{[Q^\leftarrow]}(\beta)$ for all $\beta \in \Phi^+$.

Theorem 10.3. *Let $(\bar{i}, x), (\bar{j}, y), (\bar{k}, z) \in \bar{I} \times \mathbf{k}^\times$. Then*

$$\mathrm{Hom}_{U'_q(C_n^{(1)})}(V(\varpi_{\bar{j}})_y \otimes V(\varpi_{\bar{i}})_x, V(\varpi_{\bar{k}})_z) \neq 0$$

if and only if there exists a twisted adapted class $[Q^\leftarrow]$ and $\alpha, \beta, \gamma \in \Phi^+$ such that

- (α, β) is a $[Q^\leftarrow]$ -minimal pair of γ ,
- $V(\varpi_{\bar{j}})_y = V_{[Q^\leftarrow]}(\beta)_t, \ V(\varpi_{\bar{i}})_x = V_{[Q^\leftarrow]}(\alpha)_t, \ V(\varpi_{\bar{k}})_z = V_{[Q^\leftarrow]}(\gamma)_t$ for some $t \in \mathbf{k}^\times$.

Proof. By comparing (8.16) and (10.1), our assertion is an immediate consequence of (10.2) in Definition 10.2. \square

Corollary 10.4. *The condition (b) in Definition 10.2 can be restated as follows:*

(b)' *It contains $V_{[Q^{\leftarrow}]}(\alpha_k)$ for all $\alpha_k \in \Pi$.*

APPENDIX A. TRIPLY TWISTED DYNKIN QUIVER

In [24, Appendix], the authors showed that there exist a unique \vee -foldable cluster $[\mathbf{i}_0]$ and a unique \vee^2 -foldable cluster $[\mathbf{j}_0]$ for \vee in (1.3):

$$\mathbf{i}_0 = \prod_{k=0}^5 (2 \ 1)^{k\vee} \quad \text{and} \quad \mathbf{j}_0 = \prod_{k=0}^5 (2 \ 1)^{2k\vee}.$$

In this appendix, we will introduce a triply twisted Dynkin quivers which can be analogues of twisted Dynkin quivers in Section 7.

Definition A.1. A \vee -triply twisted Dynkin quiver $Q_{(i,j,k)}^{\leftarrow}$ (resp. \vee^2 -triply twisted Dynkin quiver $Q_{(i',j',k')}^{\leftarrow}$) of D_4 is obtained by giving an orientation to each edge to following diagrams:

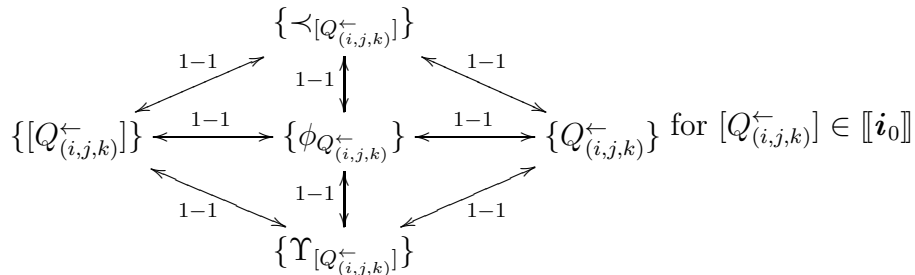
$$\begin{array}{c} \odot \\ (i,j,k) \end{array} \xrightarrow{\quad} \begin{array}{c} \circ \\ 2 \end{array} \quad \text{and} \quad \begin{array}{c} \circledast \\ (i',j',k') \end{array} \xrightarrow{\quad} \begin{array}{c} \circ \\ 2 \end{array}.$$

where $\{i, i^\vee = j, i^{2\vee} = k\} = \{1, 3, 4\} = \{i', i'^{2\vee} = j', i'^{4\vee} = k'\}$.

Then we can complete analogues of (7.3) for \vee and \vee^2 by applying the similar arguments in Section 7:

- For each triply twisted adapted class $[\mathbf{i}'_0]$, there exists a unique $Q_{(i,j,k)}^{\leftarrow}$ or $Q_{(i',j',k')}^{\leftarrow}$, the unique expression \mathbf{i}'_0 in $[\mathbf{i}'_0]$ is *adapted* to the triply twisted Dynkin quiver.
- For each triply twisted adapted class $[\mathbf{i}'_0]$, there exists a unique triply twisted Coxeter element $a \ b \ \vee = \phi_{Q_{(i,j,k)}^{\leftarrow}} \vee$ or $a \ b \ \vee^2 = \phi_{Q_{(i',j',k')}^{\leftarrow}} \vee^2$ such that

$$[\mathbf{i}'_0] = \prod_{k=0}^5 (a \ b)^{k\vee} \quad \text{or} \quad [\mathbf{i}'_0] = \prod_{k=0}^5 (a \ b)^{2k\vee}.$$



$$\begin{array}{ccccc}
& & \{ \prec [Q_{(i',j',k')}^{\leftarrow}] \} & & \\
& \swarrow 1-1 & \uparrow 1-1 & \nwarrow 1-1 & \\
\{ [Q_{(i',j',k')}^{\leftarrow}] \} & \xleftrightarrow{1-1} & \{ \phi_{Q_{(i',j',k')}^{\leftarrow}} \} & \xleftrightarrow{1-1} & \{ Q_{(i',j',k')}^{\leftarrow} \} \text{ for } [Q_{(i',j',k')}^{\leftarrow}] \in \llbracket j_0 \rrbracket \\
& \searrow 1-1 & \downarrow 1-1 & \swarrow 1-1 & \\
& & \{ \Upsilon_{[Q_{(i',j',k')}^{\leftarrow}]} \} & &
\end{array}$$

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